

A PROOF OF DE CONCINI–KAC–PROCESI CONJECTURE I. REPRESENTATIONS OF QUANTUM GROUPS AT ROOTS OF UNITY AND Q-W ALGEBRAS

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ABSTRACT. Let $U_\varepsilon(\mathfrak{g})$ be the standard simply-connected version of the Drinfeld–Jumbo quantum group at an odd m -th root of unity ε . De Concini, Kac and Procesi observed that isomorphism classes of irreducible representations of $U_\varepsilon(\mathfrak{g})$ are parameterized by the conjugacy classes in the connected simply connected algebraic group G corresponding to the simple complex Lie algebra \mathfrak{g} . They also conjectured that the dimension of a representation corresponding to a conjugacy class \mathcal{O} is divisible by $m^{\frac{1}{2}\dim \mathcal{O}}$. We show that if \mathcal{O} intersects one of special transversal slices Σ_s to the set of conjugacy classes in G then the dimension of every finite-dimensional irreducible representation of $U_\varepsilon(\mathfrak{g})$ corresponding to \mathcal{O} is divisible by $m^{\frac{1}{2}\operatorname{codim} \Sigma_s}$. This reduces the De Concini–Kac–Procesi conjecture to constructing appropriate transversal slices Σ_s such that $\dim \mathcal{O} = \operatorname{codim} \Sigma_s$ for conjugacy classes \mathcal{O} of exceptional elements in G intersecting Σ_s . Our result also implies an equivalence between a category of finite-dimensional $U_\varepsilon(\mathfrak{g})$ -modules and a category of finite-dimensional representations of a q-W algebra which can be regarded as a truncation of the quantized algebra of regular functions on Σ_s .

*To Michael Arsenyevich Semenov-Tian-Shansky
on the occasion of his 65th birthday.*

1. INTRODUCTION

It is very well known that the number of simple modules for a finite-dimensional algebra over an algebraically closed field is finite. However, often it is very difficult to classify such representations. In some important particular examples even dimensions of simple modules over finite-dimensional algebras are not known.

One of the important examples of that kind is representation theory of semisimple Lie algebras over algebraically closed fields of prime characteristic. Let \mathfrak{g}' be the Lie algebra of a semisimple algebraic group G' over an algebraically closed field \mathbf{k} of characteristic $p > h$, where h is the Coxeter number of the corresponding root system. Let $x \mapsto x^{[p]}$ be the p th power map of \mathfrak{g}' into itself. The structure of the enveloping algebra of \mathfrak{g}' is quite different from the zero characteristic case. Namely, the elements $x^p - x^{[p]}$, $x \in \mathfrak{g}'$ are central. For any linear form θ on \mathfrak{g}' , let U_θ be the quotient of the enveloping algebra of \mathfrak{g}' by the ideal generated by the central elements $x^p - x^{[p]} - \theta(x)^p$ with $x \in \mathfrak{g}'$. Then U_θ is a finite-dimensional algebra. Kac and Weisfeiler proved that any simple \mathfrak{g}' -module can be regarded as a module over U_θ for a unique θ as above (this explains why all simple \mathfrak{g}' -modules are finite-dimensional). The Kac–Weisfeiler conjecture formulated in [21] and proved in [28] says that if the G' -coadjoint orbit of θ has dimension d then $p^{\frac{d}{2}}$ divides the dimension of every finite-dimensional U_θ -module.

One can identify θ with an element of \mathfrak{g}' via the Killing form and reduce the proof of the Kac–Weisfeiler conjecture to the case of nilpotent θ . In that case Premet defines a subalgebra $U_\theta(\mathfrak{m}_\theta) \subset$

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U_θ generated by a Lie subalgebra $\mathfrak{m}_\theta \subset \mathfrak{g}'$ such that $U_\theta(\mathfrak{m}_\theta)$ has dimension $p^{\frac{d}{2}}$ and every finite-dimensional U_θ -module is $U_\theta(\mathfrak{m}_\theta)$ -free. Verification of the latter fact uses the theory of support varieties (see [16, 17, 18]). Namely, according to the theory of support varieties, in order to prove that a U_θ -module is $U_\theta(\mathfrak{m}_\theta)$ -free one should check that it is free over every subalgebra $U_\theta(x)$ generated in $U_\theta(\mathfrak{m}_\theta)$ by a single element $x \in \mathfrak{m}_\theta$.

There is a more elementary and straightforward proof of the Kac–Weisfeiler conjecture given in [27]. Another proof of the conjecture using localization of \mathcal{D} -modules is presented in [3].

Another important example of finite-dimensional algebras is related to the theory of quantum groups at roots of unity. Let \mathfrak{g} be a complex finite-dimensional semisimple Lie algebra. A remarkable property of the standard Drinfeld–Jimbo quantum group $U_q(\mathfrak{g})$ associated to \mathfrak{g} , where q is a primitive m -th root of unity, is that its center contains a huge commutative subalgebra isomorphic to the algebra Z_G of regular functions on (a finite covering of a big cell in) a complex algebraic group G with Lie algebra \mathfrak{g} . In this paper we consider the simply-connected version of $U_q(\mathfrak{g})$ and the case when m is odd. In that case G is the connected, simply connected algebraic group corresponding to \mathfrak{g} .

Consider finite-dimensional representations of $U_q(\mathfrak{g})$, on which Z_G acts according to nontrivial characters η_g given by evaluation of regular functions at various points $g \in G$. Note that all irreducible representations of $U_q(\mathfrak{g})$ are of that kind, and every such representation is a representation of the algebra $U_{\eta_g} = U_q(\mathfrak{g})/U_q(\mathfrak{g})\text{Ker } \eta_g$ for some η_g . In [11] De Concini, Kac and Procesi showed that if g_1 and g_2 are two conjugate elements of G then the algebras $U_{\eta_{g_1}}$ and $U_{\eta_{g_2}}$ are isomorphic. Moreover in [11] De Concini, Kac and Procesi formulated the following conjecture.

De Concini–Kac–Procesi conjecture. *The dimension of any finite-dimensional representation of the algebra U_{η_g} is divisible by $m^{\frac{1}{2}\dim \mathcal{O}_g}$, where \mathcal{O}_g is the conjugacy class of g .*

This conjecture is the quantum group counterpart of the Kac–Weisfeiler conjecture for semisimple Lie algebras over fields of prime characteristic.

As it is shown in [12] it suffices to verify the De Concini–Kac–Procesi conjecture in case of exceptional elements $g \in G$ (an element $g \in G$ is called exceptional if its centralizer in G has a finite center). However, the De Concini–Kac–Procesi conjecture is related to the geometry of the group G which is much more complicated than the geometry of the linear space \mathfrak{g}' in case of the Kac–Weisfeiler conjecture.

The De Concini–Kac–Procesi conjecture is known to be true for the conjugacy classes of regular elements ([13]), for the subregular unipotent conjugacy classes in type A_n when m is a power of a prime ([5]), for all conjugacy classes in A_n when m is a prime ([7]), for the conjugacy classes \mathcal{O}_g of $g \in SL_n$ when the conjugacy class of the unipotent part of g is spherical ([6]), and for spherical conjugacy classes ([4]). In [24] a proof of the De Concini–Kac–Procesi using localization of quantum \mathcal{D} -modules is outlined in case of unipotent conjugacy classes.

In this paper following Premet’s philosophy we construct certain subalgebras $U_{\eta_g}(\mathfrak{m}_+)$ in U_{η_g} over which U_{η_g} -modules are free, at least for some $g \in G$. Since the De Concini–Kac–Procesi conjecture is related to the structure of conjugacy classes in G it is natural to look at transversal slices to the set of conjugacy classes. It turns out that the definition of the subalgebras $U_{\eta_g}(\mathfrak{m}_+)$ is related to the existence of some special transversal slices Σ_s to the set of conjugacy classes in G . These slices Σ_s associated to (conjugacy classes of) elements s in the Weyl group of \mathfrak{g} were introduced by the author in [32]. The slices Σ_s play the role of Slodowy slices in algebraic group theory. In the particular case of elliptic Weyl group elements these slices were also introduced later by He and Lusztig in paper [20] within a different framework.

A remarkable property of a slice Σ_s is that if g is conjugate to an element in Σ_s then U_{η_g} has a subalgebra of dimension $m^{\frac{1}{2}\text{codim } \Sigma_s}$ with a nontrivial character. If $g \in \Sigma_s$ (in fact g may belong

to a larger variety) then the corresponding subalgebra $U_{\eta_g}(\mathfrak{m}_+)$ can be explicitly described in terms of quantum group analogues of root vectors. There are also analogues of subalgebras $U_{\eta_g}(\mathfrak{m}_+)$ in $U_q(\mathfrak{g})$ in case of generic q (see [33]).

In Section 9 we prove, in particular, that if $g \in \Sigma_s$ then every finite-dimensional U_{η_g} -module is free over a subalgebra $\tilde{U}_{\eta_g}(\mathfrak{m}_+)$ isomorphic to $U_{\eta_g}(\mathfrak{m}_+)$. Thus the dimension of every such module is divisible by $m^{\frac{1}{2}\text{codim } \Sigma_s}$, and if the conjugacy class of g intersects Σ_s strictly transversally in the sense that $\text{codim } \Sigma_s = \dim \mathcal{O}_g$, this proves the De Concini–Kac–Procesi conjecture. Thus the De Concini–Kac–Procesi conjecture is reduced to constructing appropriate transversal slices Σ_s such that $\dim \mathcal{O}_g = \text{codim } \Sigma_s$ for conjugacy classes \mathcal{O}_g of exceptional elements in G . This will be proved in a subsequent paper. In Section 9 it is also shown that the rank of every finite-dimensional U_{η_g} -module V over $\tilde{U}_{\eta_g}(\mathfrak{m}_+)$ is equal to the dimension of the space V_χ of the so-called Whittaker vectors in V , which consists of elements $v \in V$ such that $xv = \chi(x)v$, $x \in \tilde{U}_{\eta_g}(\mathfrak{m}_+)$, and χ is a nontrivial character of $\tilde{U}_{\eta_g}(\mathfrak{m}_+)$. Whittaker vectors are studied in detail in Section 8.

The proof of the main statement of Section 9 is reduced to the fact that for certain g every finite-dimensional U_{η_g} -module V is free over every subalgebra $U_{\eta_g}(f)$ in $U_{\eta_g}(\mathfrak{m}_+)$ generated by a quantum analogue f of a root vector in a Lie subalgebra $\mathfrak{m}_+ \subset \mathfrak{g}$. The support variety technique can not be transferred to the case of quantum groups straightforwardly. The notion of the support variety is still available in case of quantum groups (see [15, 19, 25]). But in practical application it is much less efficient since in case of quantum groups there is no underlying linear space. However, one can show that V is free over $U_{\eta_g}(\mathfrak{m}_+)$ using a complicated induction over appropriately ordered set of root vectors in \mathfrak{m}_+ . In case of restricted representations of a small quantum group this was done in [15]. The situation in [15] is rather similar to the case of the trivial character η_1 corresponding to the identity element $1 \in G$. In the case considered in this paper the induction is even more complicated because the algebra $U_{\eta_g}(\mathfrak{m}_+)$ has the Jacobson radical \mathcal{J} (see Section 8), and the quotient $U_{\eta_g}(\mathfrak{m}_+)/\mathcal{J}$ is a nontrivial semisimple algebra. This shows a major difference between Lie algebras and quantum groups: in case of Lie algebras \mathfrak{g}' over fields of prime characteristic the algebras $U_\theta(\mathfrak{m}_\theta)$ are local while in the quantum group case the algebras $U_{\eta_g}(\mathfrak{m}_+)$, which play the role of $U_\theta(\mathfrak{m}_\theta)$, are not local.

Slices Σ_s also appear in Section 10 in a different incarnation. Namely, we show that for g conjugate to an element in Σ_s the category of finite-dimensional U_{η_g} -modules is equivalent to a category of finite-dimensional modules over an algebra $W_q^s(G)$ which can be regarded as a noncommutative deformation of a truncated version of the algebra of regular functions on Σ_s . In case of generic q such algebras, called q-W algebras, were introduced and studied in [33]. In fact U_{η_g} is the algebra of matrices of size $m^{\frac{1}{2}\text{codim } \Sigma_s}$ over the algebra $W_q^s(G)$ which has dimension $m^{\frac{1}{2}\dim \Sigma_s}$. In case of Lie algebras over fields of prime characteristic similar results were obtained in [29].

The proofs of statements in Sections 8, 9 and 10 require some preliminary results which are presented in Sections 2–7.

2. NOTATION

Fix the notation used throughout of the text. Let G be a connected simply connected finite-dimensional complex simple Lie group, \mathfrak{g} its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. Let α_i , $i = 1, \dots, l$, $l = \text{rank}(\mathfrak{g})$ be a system of simple roots, $\Delta_+ = \{\beta_1, \dots, \beta_N\}$ the set of positive roots. Let H_1, \dots, H_l be the set of simple root generators of \mathfrak{h} .

Let a_{ij} be the corresponding Cartan matrix, and let d_1, \dots, d_l be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique non-degenerate invariant symmetric bilinear form $(,)$ on \mathfrak{g} such that $(H_i, H_j) = d_j^{-1} a_{ij}$. It induces an isomorphism of vector spaces $\mathfrak{h} \simeq \mathfrak{h}^*$

under which $\alpha_i \in \mathfrak{h}^*$ corresponds to $d_i H_i \in \mathfrak{h}$. We denote by α^\vee the element of \mathfrak{h} that corresponds to $\alpha \in \mathfrak{h}^*$ under this isomorphism. The induced bilinear form on \mathfrak{h}^* is given by $(\alpha_i, \alpha_j) = b_{ij}$.

Let W be the Weyl group of the root system Δ . W is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections s_1, \dots, s_l ,

$$s_i(h) = h - \alpha_i(h)H_i, \quad h \in \mathfrak{h}.$$

The action of W preserves the bilinear form $(,)$ on \mathfrak{h} . We denote a representative of $w \in W$ in G by the same letter. For $w \in W, g \in G$ we write $w(g) = wgw^{-1}$. For any root $\alpha \in \Delta$ we also denote by s_α the corresponding reflection.

Let \mathfrak{b}_+ be the positive Borel subalgebra and \mathfrak{b}_- the opposite Borel subalgebra; let $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ be their nilradicals. Let $H = \exp \mathfrak{h}, N_+ = \exp \mathfrak{n}_+, N_- = \exp \mathfrak{n}_-, B_+ = HN_+, B_- = HN_-$ be the Cartan subgroup, the maximal unipotent subgroups and the Borel subgroups of G which correspond to the Lie subalgebras $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{b}_+$ and \mathfrak{b}_- , respectively.

We identify \mathfrak{g} and its dual by means of the canonical invariant bilinear form. Then the coadjoint action of G on \mathfrak{g}^* is naturally identified with the adjoint one. We also identify $\mathfrak{n}_+^* \cong \mathfrak{n}_-, \mathfrak{b}_+^* \cong \mathfrak{b}_-$.

Let \mathfrak{g}_β be the root subspace corresponding to a root $\beta \in \Delta$, $\mathfrak{g}_\beta = \{x \in \mathfrak{g} | [h, x] = \beta(h)x \text{ for every } h \in \mathfrak{h}\}$. $\mathfrak{g}_\beta \subset \mathfrak{g}$ is a one-dimensional subspace. It is well-known that for $\alpha \neq -\beta$ the root subspaces \mathfrak{g}_α and \mathfrak{g}_β are orthogonal with respect to the canonical invariant bilinear form. Moreover \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$ are non-degenerately paired by this form.

Root vectors $X_\alpha \in \mathfrak{g}_\alpha$ satisfy the following relations:

$$[X_\alpha, X_{-\alpha}] = (X_\alpha, X_{-\alpha})\alpha^\vee.$$

Note also that in this paper we denote by \mathbb{N} the set of nonnegative integer numbers, $\mathbb{N} = \{0, 1, \dots\}$.

3. QUANTUM GROUPS

The standard simply connected quantum group $U_q(\mathfrak{g})$ associated to a complex finite-dimensional simple Lie algebra \mathfrak{g} is the algebra over $\mathbb{C}(q)$ generated by elements $L_i, L_i^{-1}, X_i^+, X_i^-, i = 1, \dots, l$, and with the following defining relations:

$$(3.1) \quad \begin{aligned} [L_i, L_j] &= 0, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_i X_j^\pm L_i^{-1} = q_i^{\pm \delta_{ij}} X_j^\pm, \\ X_i^+ X_j^- - X_j^- X_i^+ &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \end{aligned}$$

$$\text{where } K_i = \prod_{j=1}^l L_j^{a_{ji}}, \quad q_i = q^{d_i},$$

and the quantum Serre relations:

$$(3.2) \quad \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} (X_i^\pm)^{1-a_{ij}-r} X_j^\pm (X_i^\pm)^r = 0, \quad i \neq j,$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q! [n-m]_q!}, \quad [n]_q! = [n]_q \dots [1]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

$U_q(\mathfrak{g})$ is a Hopf algebra with comultiplication defined by

$$\Delta(L_i^{\pm 1}) = L_i^{\pm 1} \otimes L_i^{\pm 1},$$

$$\Delta(X_i^\pm) = X_i^\pm \otimes K_i + 1 \otimes X_i^\pm,$$

$$\Delta(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-,$$

antipode defined by

$$S(L_i^{\pm 1}) = L_i^{\mp 1}, \quad S(X_i^+) = -X_i^+ K_i^{-1}, \quad S(X_i^-) = -K_i X_i^-,$$

and counit defined by

$$\varepsilon(L_i^{\pm 1}) = 1, \quad \varepsilon(X_i^{\pm}) = 0.$$

Now we shall explicitly describe a basis for $U_q(\mathfrak{g})$. First following [9] we recall the construction of root vectors of $U_q(\mathfrak{g})$ in terms of a braid group action on $U_q(\mathfrak{g})$. Let m_{ij} , $i \neq j$ be equal to 2, 3, 4, 6 if $a_{ij}a_{ji}$ is equal to 0, 1, 2, 3. The braid group $\mathcal{B}_{\mathfrak{g}}$ associated to \mathfrak{g} has generators T_i , $i = 1, \dots, l$, and defining relations

$$T_i T_j T_i T_j \dots = T_j T_i T_j T_i \dots$$

for all $i \neq j$, where there are m_{ij} T 's on each side of the equation.

There is an action of the braid group $\mathcal{B}_{\mathfrak{g}}$ by algebra automorphisms of $U_q(\mathfrak{g})$ defined on the standard generators as follows:

$$T_i(X_i^+) = -X_i^- K_i, \quad T_i(X_i^-) = -K_i^{-1} X_i^+, \quad T_i(L_j) = L_j K_i^{-1},$$

$$T_i(X_j^+) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} (X_i^+)^{(-a_{ij}-r)} X_j^+ (X_i^+)^{(r)}, \quad i \neq j,$$

$$T_i(X_j^-) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^r (X_i^-)^{(r)} X_j^- (X_i^-)^{(-a_{ij}-r)}, \quad i \neq j,$$

where

$$(X_i^+)^{(r)} = \frac{(X_i^+)^r}{[r]_{q_i}!}, \quad (X_i^-)^{(r)} = \frac{(X_i^-)^r}{[r]_{q_i}!}, \quad r \geq 0, \quad i = 1, \dots, l.$$

Recall that an ordering of a set of positive roots Δ_+ is called normal if all simple roots are written in an arbitrary order, and for any three roots α, β, γ such that $\gamma = \alpha + \beta$ we have either $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$.

Any two normal orderings in Δ_+ can be reduced to each other by the so-called elementary transpositions (see [34], Theorem 1). The elementary transpositions for rank 2 root systems are inversions of the following normal orderings (or the inverse normal orderings):

$$(3.3) \quad \begin{array}{ll} \alpha, \beta & A_1 + A_1 \\ \alpha, \alpha + \beta, \beta & A_2 \\ \alpha, \alpha + \beta, \alpha + 2\beta, \beta & B_2 \\ \alpha, \alpha + \beta, 2\alpha + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta & G_2 \end{array}$$

where it is assumed that $(\alpha, \alpha) \geq (\beta, \beta)$. Moreover, any normal ordering in a rank 2 root system is one of orderings (3.3) or one of the inverse orderings.

In general an elementary inversion of a normal ordering in a set of positive roots Δ_+ is the inversion of an ordered segment of form (3.3) (or of a segment with the inverse ordering) in the ordered set Δ_+ , where $\alpha - \beta \notin \Delta$.

For any reduced decomposition $w_0 = s_{i_1} \dots s_{i_D}$ of the longest element w_0 of the Weyl group W of \mathfrak{g} the set

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \dots, \beta_D = s_{i_1} \dots s_{i_{D-1}} \alpha_{i_D}$$

is a normal ordering in Δ_+ , and there is one to one correspondence between normal orderings of Δ_+ and reduced decompositions of w_0 (see [35]).

Now fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_D}$ of the longest element w_0 of the Weyl group W of \mathfrak{g} and define the corresponding root vectors in $U_q(\mathfrak{g})$ by

$$(3.4) \quad X_{\beta_k}^{\pm} = T_{i_1} \dots T_{i_{k-1}} X_{i_k}^{\pm}.$$

Proposition 3.1. *For $\beta = \sum_{i=1}^l m_i \alpha_i$, $m_i \in \mathbb{N}$ X_{β}^{\pm} is a polynomial in the noncommutative variables X_i^{\pm} homogeneous in each X_i^{\pm} of degree m_i .*

Note that one can construct root vectors in the Lie algebra \mathfrak{g} in a similar way. Namely, if $X_{\pm\alpha_i}$ are simple root vectors of \mathfrak{g} then one can define an action of the braid group $\mathcal{B}_{\mathfrak{g}}$ by algebra automorphisms of \mathfrak{g} defined on the standard generators as follows:

$$T_i(X_{\pm\alpha_i}) = -X_{\mp\alpha_i}, \quad T_i(H_j) = H_j - a_{ji}H_i,$$

$$T_i(X_{\alpha_j}) = \frac{1}{(-a_{ij})!} \text{ad}_{X_{\alpha_i}}^{-a_{ij}} X_{\alpha_j}, \quad i \neq j,$$

$$T_i(X_{-\alpha_j}) = \frac{(-1)^{a_{ij}}}{(-a_{ij})!} \text{ad}_{X_{-\alpha_i}}^{-a_{ij}} X_{-\alpha_j}, \quad i \neq j.$$

Now the root vectors $X_{\pm\beta_k} \in \mathfrak{g}_{\pm\beta_k}$ of \mathfrak{g} can be defined by

$$(3.5) \quad X_{\pm\beta_k} = T_{i_1} \dots T_{i_{k-1}} X_{\pm\alpha_{i_k}}.$$

The root vectors X_{β}^{-} satisfy the following relations:

$$(3.6) \quad X_{\alpha}^{-} X_{\beta}^{-} - q^{(\alpha, \beta)} X_{\beta}^{-} X_{\alpha}^{-} = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C(k_1, \dots, k_n) (X_{\delta_n}^{-})^{(k_n)} (X_{\delta_{n-1}}^{-})^{(k_{n-1})} \dots (X_{\delta_1}^{-})^{(k_1)}, \quad \alpha < \beta,$$

where for $\alpha \in \Delta_+$ we put $(X_{\alpha}^{\pm})^{(k)} = \frac{(X_{\alpha}^{\pm})^k}{[k]_{q_{\alpha}}!}$, $k \geq 0$, $q_{\alpha} = q^{d_i}$ if the positive root α is Weyl group conjugate to the simple root α_i , $C(k_1, \dots, k_n) \in \mathbb{C}[q, q^{-1}]$.

Let $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{n}_-)$ and $U_q(\mathfrak{h})$ be the subalgebras of $U_q(\mathfrak{g})$ generated by the X_i^+ , by the X_i^- and by the $L_i^{\pm 1}$, respectively.

Now using the root vectors X_{β}^{\pm} we can construct a basis of $U_q(\mathfrak{g})$. Define for $\mathbf{r} = (r_1, \dots, r_D) \in \mathbb{N}^D$,

$$(X^+)^{(\mathbf{r})} = (X_{\beta_1}^+)^{(r_1)} \dots (X_{\beta_D}^+)^{(r_D)},$$

$$(X^-)^{(\mathbf{r})} = (X_{\beta_D}^-)^{(r_D)} \dots (X_{\beta_1}^-)^{(r_1)},$$

and for $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$,

$$L^{\mathbf{s}} = L_1^{s_1} \dots L_l^{s_l}.$$

Proposition 3.2. ([22], **Proposition 3.3**) *The elements $(X^+)^{(\mathbf{r})}$, $(X^-)^{(\mathbf{t})}$ and $L^{\mathbf{s}}$, for $\mathbf{r}, \mathbf{t} \in \mathbb{N}^D$, $\mathbf{s} \in \mathbb{Z}^l$, form topological bases of $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{n}_-)$ and $U_q(\mathfrak{h})$, respectively, and the products $(X^+)^{(\mathbf{r})} L^{\mathbf{s}} (X^-)^{(\mathbf{t})}$ form a basis of $U_q(\mathfrak{g})$. In particular, multiplication defines an isomorphism of $\mathbb{C}(q)$ -modules:*

$$U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+) \rightarrow U_q(\mathfrak{g}).$$

Let $U_{\mathcal{A}}(\mathfrak{g})$ be the subalgebra in $U_q(\mathfrak{g})$ over the ring $\mathcal{A} = \mathbb{C}[q, q^{-1}]$ generated over \mathcal{A} by the elements $L_i^{\pm 1}$, $\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, X_i^{\pm} , $i = 1, \dots, l$. The most important for us is the specialization $U_{\varepsilon}(\mathfrak{g})$ of $U_{\mathcal{A}}(\mathfrak{g})$, $U_{\varepsilon}(\mathfrak{g}) = U_{\mathcal{A}}(\mathfrak{g}) / (q - \varepsilon) U_{\mathcal{A}}(\mathfrak{g})$, $\varepsilon \in \mathbb{C}^*$. $U_{\mathcal{A}}(\mathfrak{g})$ and $U_{\varepsilon}(\mathfrak{g})$ are Hopf algebras with the comultiplication

induced from $U_q(\mathfrak{g})$. If in addition $\varepsilon^{2d_i} \neq 1$ for $i = 1, \dots, l$ then $U_\varepsilon(\mathfrak{g})$ is generated over \mathbb{C} by $L_i^{\pm 1}$, X_i^\pm , $i = 1, \dots, l$ subject to relations (3.1) and (3.2) where $q = \varepsilon$. We also have the following obvious consequence of Proposition 3.2.

Proposition 3.3. *Let $U_\varepsilon(\mathfrak{n}_+)$, $U_\varepsilon(\mathfrak{n}_-)$ and $U_\varepsilon(\mathfrak{h})$ be the subalgebras of $U_\varepsilon(\mathfrak{g})$ generated by the X_i^+ , by the X_i^- and by the $L_i^{\pm 1}$, respectively. The elements $(X^+)^{\mathbf{r}} = (X_{\beta_1}^+)^{r_1} \dots (X_{\beta_D}^+)^{r_D}$, $(X^-)^{\mathbf{t}} = (X_{\beta_D}^-)^{t_D} \dots (X_{\beta_1}^-)^{t_1}$ and $L^{\mathbf{s}}$, for $\mathbf{r}, \mathbf{t} \in \mathbb{N}^N$, $\mathbf{s} \in \mathbb{Z}^l$, form bases of $U_\varepsilon(\mathfrak{n}_+)$, $U_\varepsilon(\mathfrak{n}_-)$ and $U_\varepsilon(\mathfrak{h})$, respectively, and the products $(X^+)^{\mathbf{r}} L^{\mathbf{s}} (X^-)^{\mathbf{t}}$ form a basis of $U_\varepsilon(\mathfrak{g})$. In particular, multiplication defines an isomorphism of vector spaces:*

$$U_\varepsilon(\mathfrak{n}_-) \otimes U_\varepsilon(\mathfrak{h}) \otimes U_\varepsilon(\mathfrak{n}_+) \rightarrow U_\varepsilon(\mathfrak{g}).$$

The root vectors X_β^+ satisfy the following relations in $U_\varepsilon(\mathfrak{g})$:

$$(3.7) \quad X_\alpha^- X_\beta^- - \varepsilon^{(\alpha, \beta)} X_\beta^- X_\alpha^- = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C(k_1, \dots, k_n) (X_{\delta_n}^-)^{(k_n)} (X_{\delta_{n-1}}^-)^{(k_{n-1})} \dots (X_{\delta_1}^-)^{(k_1)}, \quad \alpha < \beta,$$

where $C(k_1, \dots, k_n) \in \mathbb{C}$.

4. QUANTUM GROUPS AT ROOTS OF UNITY

Let m be a an odd positive integer number, and $m > d_i$ for all i , ε a primitive m -th root of unity. In this section, following [9], Section 9.2, we recall some results on the structure of the algebra $U_\varepsilon(\mathfrak{g})$. We keep the notation introduced in Section 2.

Let Z_ε be the center of $U_\varepsilon(\mathfrak{g})$.

Proposition 4.1. ([10], Corollary 3.3, [11], Theorems 3.5, 7.6 and Proposition 4.5) *Fix the normal ordering in the positive root system Δ_+ corresponding a reduced decomposition $w_0 = s_{i_1} \dots s_{i_D}$ of the longest element w_0 of the Weyl group W of \mathfrak{g} and let X_α^\pm be the corresponding root vectors in $U_\varepsilon(\mathfrak{g})$, and $X_{\pm \beta_k}$ the corresponding root vectors in \mathfrak{g} . Let $x_\alpha^- = (\varepsilon_\alpha - \varepsilon_\alpha^{-1})^m (X_\alpha^-)^m$, $x_\alpha^+ = (\varepsilon_\alpha - \varepsilon_\alpha^{-1})^m T_0 (X_\alpha^-)^m$, where $T_0 = T_{i_1} \dots T_{i_D}$, $\alpha \in \Delta_+$ and $l_i = L_i^m$, $i = 1, \dots, l$ be elements of $U_\varepsilon(\mathfrak{g})$.*

Then the following statements are true.

- (i) *The elements x_α^\pm , $\alpha \in \Delta_+$, l_i , $i = 1, \dots, l$ lie in Z_ε .*
- (ii) *Let Z_0 (Z_0^\pm and Z_0^0) be the subalgebras of Z_ε generated by the x_α^\pm and the $l_i^{\pm 1}$ (respectively by the x_α^\pm and by the $l_i^{\pm 1}$). Then $Z_0^\pm \subset U_\varepsilon(\mathfrak{n}_\pm)$, $Z_0^0 \subset U_\varepsilon(\mathfrak{h})$, Z_0^\pm is the polynomial algebra with generators x_α^\pm , Z_0^0 is the algebra of Laurent polynomials in the l_i , $Z_0^\pm = U_\varepsilon(\mathfrak{n}_\pm) \cap Z_0$, and multiplication defines an isomorphism of algebras*

$$Z_0^- \otimes Z_0^0 \otimes Z_0^+ \rightarrow Z_0.$$

- (iii) *$U_\varepsilon(\mathfrak{g})$ is a free Z_0 -module with basis the set of monomials $(X^+)^{\mathbf{r}} L^{\mathbf{s}} (X^-)^{\mathbf{t}}$ in the statement of Proposition 3.3 for which $0 \leq r_k, t_k, s_i < m$ for $i = 1, \dots, l$, $k = 1, \dots, D$.*

- (iv) *$\text{Spec}(Z_0) = \mathbb{C}^{2D} \times (\mathbb{C}^*)^l$ is a complex affine space of dimension equal to $\dim \mathfrak{g}$, $\text{Spec}(Z_\varepsilon)$ is a normal affine variety and the map*

$$\tau : \text{Spec}(Z_\varepsilon) \rightarrow \text{Spec}(Z_0)$$

induced by the inclusion $Z_0 \hookrightarrow Z_\varepsilon$ is a finite map of degree m^l .

- (v) *The subalgebra Z_0 is preserved by the action of the braid group automorphisms T_i .*

- (vi) *Let G be the connected simply connected Lie group corresponding to the Lie algebra \mathfrak{g} and G_0^* the solvable algebraic subgroup in $G \times G$ which consists of elements of the form $(L'_+, L'_-) \in G \times G$,*

$$(L'_+, L'_-) = (t, t^{-1})(n'_+, n'_-), \quad n'_\pm \in N_\pm, \quad t \in H.$$

Then $\text{Spec}(Z_0^0)$ can be naturally identified with the maximal torus H in G , and the map

$$\tilde{\pi} : \text{Spec}(Z_0) = \text{Spec}(Z_0^+) \times \text{Spec}(Z_0^0) \times \text{Spec}(Z_0^-) \rightarrow G_0^*,$$

$$\tilde{\pi}(u_+, t, u_-) = (t\mathbf{X}^+(u_+), t^{-1}\mathbf{X}^-(u_-)^{-1}), \quad u_{\pm} \in \text{Spec}(Z_0^{\pm}), \quad t \in \text{Spec}(Z_0^0),$$

$$\mathbf{X}^{\pm} : \text{Spec}(Z_0^{\pm}) \rightarrow N_{\pm}$$

$$\mathbf{X}^- = \exp(x_{\beta_D}^- X_{-\beta_D}) \exp(x_{\beta_{D-1}}^- X_{-\beta_{D-1}}) \dots \exp(x_{\beta_1}^- X_{-\beta_1}),$$

$$\mathbf{X}^+ = \exp(x_{\beta_D}^+ T_0(X_{-\beta_D})) \exp(x_{\beta_{D-1}}^+ T_0(X_{-\beta_{D-1}})) \dots \exp(x_{\beta_1}^+ T_0(X_{-\beta_1})),$$

where $x_{\beta_i}^{\pm}$ should be regarded as complex-valued functions on $\text{Spec}(Z_0)$, is an isomorphism of varieties independent of the choice of reduced decomposition of w_0 .

Remark 4.1. In fact $\text{Spec}(Z_0)$ carries a natural structure of a Poisson–Lie group, and the map $\tilde{\pi}$ is an isomorphism of algebraic Poisson–Lie groups if G_0^* is regarded as the dual Poisson–Lie group to the Poisson–Lie group G equipped with the standard Sklyanin bracket (see [11], Theorem 7.6). We shall not need this fact in this paper.

Let $\mathbf{K} : \text{Spec}(Z_0^0) \rightarrow H$ be the map defined by $\mathbf{K}(h) = h^2$, $h \in H$.

Proposition 4.2. ([11], Corollary 4.7) Let $G^0 = N_- H N_+$ be the big cell in G . Then the map

$$\pi = \mathbf{X}^- \mathbf{K} \mathbf{X}^+ : \text{Spec}(Z_0) \rightarrow G^0$$

is independent of the choice of reduced decomposition of w_0 , and is an unramified covering of degree 2^l .

Define derivations \underline{x}_i^{\pm} of $U_{\mathcal{A}}(\mathfrak{g})$ by

$$(4.1) \quad \underline{x}_i^+(u) = \left[\frac{(X_i^+)^m}{[m]_{q_i}!}, u \right], \quad \underline{x}_i^-(u) = T_0 \underline{x}_i^+ T_0^{-1}(u), \quad i = 1, \dots, l.$$

Let \widehat{Z}_0 be the algebra of formal power series in the x_{α}^{\pm} , $\alpha \in \Delta_+$, and the $l_i^{\pm 1}$, $i = 1, \dots, l$, which define holomorphic functions on $\text{Spec}(Z_0) = \mathbb{C}^{2D} \times (\mathbb{C}^*)^l$. Let

$$\widehat{U}_{\varepsilon}(\mathfrak{g}) = U_{\varepsilon}(\mathfrak{g}) \otimes_{Z_0} \widehat{Z}_0, \quad \widehat{Z}_{\varepsilon} = Z_{\varepsilon} \otimes_{Z_0} \widehat{Z}_0.$$

Proposition 4.3. ([10], Propositions 3.4, 3.5, [11], Proposition 6.1)

(i) On specializing to $q = \varepsilon$, (4.1) induces a well-defined derivation \underline{x}_i^{\pm} of $U_{\varepsilon}(\mathfrak{g})$.

(ii) The series

$$\exp(t \underline{x}_i^{\pm}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\underline{x}_i^{\pm})^k$$

converge for all $t \in \mathbb{C}$ to a well-defined automorphisms of the algebra $\widehat{U}_{\varepsilon}(\mathfrak{g})$.

(iii) Let \mathcal{G} be the group of automorphisms generated by the one-parameter groups $\exp(t \underline{x}_i^{\pm})$, $i = 1, \dots, l$. The action of \mathcal{G} on $\widehat{U}_{\varepsilon}(\mathfrak{g})$ preserves the subalgebras $\widehat{Z}_{\varepsilon}$ and \widehat{Z}_0 , and hence acts by holomorphic automorphisms on the complex algebraic varieties $\text{Spec}(Z_{\varepsilon})$ and $\text{Spec}(Z_0)$.

(iv) Let \mathcal{O} be a conjugacy class in G . The intersection $\mathcal{O}^0 = \mathcal{O} \cap G^0$ is a smooth connected variety, and the connected components of the variety $\pi^{-1}(\mathcal{O}^0)$ are \mathcal{G} -orbits in $\text{Spec}(Z_0)$.

(v) If \mathcal{P} is a \mathcal{G} -orbit in $\text{Spec}(Z_0)$ then the connected components of $\tau^{-1}(\mathcal{P})$ are \mathcal{G} -orbits in $\text{Spec}(Z_{\varepsilon})$.

Given a homomorphism $\eta : Z_0 \rightarrow \mathbb{C}$, let

$$U_\eta(\mathfrak{g}) = U_\varepsilon(\mathfrak{g})/I_\eta,$$

where I_η is the ideal in $U_\varepsilon(\mathfrak{g})$ generated by elements $z - \eta(z)$, $z \in Z_0$. By part (iii) of Proposition 4.1 $U_\eta(\mathfrak{g})$ is an algebra of dimension $m^{\dim \mathfrak{g}}$ with basis the set of monomials $(X^+)^{\mathbf{r}} L^{\mathbf{s}} (X^-)^{\mathbf{t}}$ in the statement of Proposition 3.3 for which $0 \leq r_k, t_k, s_i < m$ for $i = 1, \dots, l$, $k = 1, \dots, D$.

If V is an irreducible finite-dimensional representation of $U_\varepsilon(\mathfrak{g})$ then by the Schur lemma $zv = \theta(z)v$ for all $v \in V$ and $z \in Z_\varepsilon$ and some character $\theta : Z_\varepsilon \rightarrow \mathbb{C}$. Therefore we get a natural map

$$X : \text{Rep}(U_\varepsilon(\mathfrak{g})) \rightarrow \text{Spec}(Z_\varepsilon),$$

where $\text{Rep}(U_\varepsilon(\mathfrak{g}))$ is the set of equivalence classes of irreducible finite-dimensional representations of $U_\varepsilon(\mathfrak{g})$, and V is in fact a representation of the algebra $U_\eta(\mathfrak{g})$ for $\eta = X(\theta)$. We shall identify this representation with V . Observe that every finite-dimensional irreducible representation in $\text{Rep}(U_\varepsilon(\mathfrak{g}))$ is a representation of $U_\eta(\mathfrak{g})$ for some $\eta \in \text{Spec}(Z_0)$.

If $\tilde{g} \in \mathcal{G}$ then for any $\eta \in \text{Spec}(Z_0)$ we have $\tilde{g}\eta \in \text{Spec}(Z_0)$ by part (iii) of Proposition 4.3, and by part (ii) of the same proposition \tilde{g} induces an isomorphism of algebras,

$$\tilde{g} : U_\eta(\mathfrak{g}) \rightarrow U_{\tilde{g}\eta}(\mathfrak{g}).$$

This establishes a bijection between the sets $\text{Rep}(U_\eta(\mathfrak{g}))$ and $\text{Rep}(U_{\tilde{g}\eta}(\mathfrak{g}))$ of equivalence classes of irreducible finite-dimensional representations of $U_\eta(\mathfrak{g})$ and $U_{\tilde{g}\eta}(\mathfrak{g})$,

$$(4.2) \quad \tilde{g} : \text{Rep}(U_\eta(\mathfrak{g})) \rightarrow \text{Rep}(U_{\tilde{g}\eta}(\mathfrak{g})).$$

For every finite-dimensional representation V of $U_\eta(\mathfrak{g})$, and $\tilde{g} \in \mathcal{G}$ we denote by $V^{\tilde{g}}$ the corresponding representation of $U_{\tilde{g}\eta}(\mathfrak{g})$.

For any element $g \in G$ let $g_s, g_u \in G$ be the semisimple and the unipotent part of g so that $g = g_s g_u$. Recall that g is called exceptional if the centralizer of g_s in G has a finite center.

Let $\varphi = \pi\tau X : \text{Rep}(U_\varepsilon(\mathfrak{g})) \rightarrow G^0$ be the composition of the three maps π, τ and X defined above. A finite-dimensional irreducible representation V of $U_\varepsilon(\mathfrak{g})$ is called exceptional if $\varphi(V) \in G^0 \subset G$ is an exceptional element.

Observe that the conjugacy class of every non-exceptional element contains an element $g \in G$ such that

$$(4.3) \quad g_s \in H, \quad g_u \in N_-,$$

$$(4.4) \quad \text{the Lie algebra } \mathfrak{h}_g \text{ of the center of the centralizer of } g_s \text{ in } G \text{ is nontrivial,}$$

and

$$(4.5) \quad \Delta' = \{\alpha \in \Delta : \alpha|_{\mathfrak{h}_g} = 0\} = \mathbb{Z}\Gamma' \cap \Delta,$$

where $\Gamma' \subset \Gamma$ is a proper subset of the set of simple positive roots Γ .

Therefore if V is a non-exceptional irreducible finite-dimensional representation of $U_\varepsilon(\mathfrak{g})$ then V can be regarded as a representation of the algebra $U_\eta(\mathfrak{g})$ for $\eta = \tau X(V)$, and by part (iv) of Proposition 4.3 there exists an element $\tilde{g} \in \mathcal{G}$ such that $\pi(\tilde{g}\eta)$ satisfies properties (4.3)-(4.5), and by (4.2) $\tilde{g}V$ can be regarded as a representation of the algebra $U_{\tilde{g}\eta}(\mathfrak{g})$.

Replacing V with $\tilde{g}V$ we may assume that V is an irreducible representation of the algebra $U_\eta(\mathfrak{g})$ such that $g = \pi(\eta)$ satisfies (4.3)-(4.5).

Let $U'_\varepsilon(\mathfrak{g})$ be the subalgebra of $U_\varepsilon(\mathfrak{g})$ generated by $U_\varepsilon(\mathfrak{h})$ and all the elements X_i^\pm such that $\alpha_i \in \Gamma'$. Denote by $U'_\eta(\mathfrak{g})$ the quotient of $U'_\varepsilon(\mathfrak{g})$ by the ideal generated by elements $z - \eta(z)$, $z \in Z_0 \cap U'_\varepsilon(\mathfrak{g})$. Now let $U_\varepsilon^g(\mathfrak{g}) = U'_\varepsilon(\mathfrak{g})U_\varepsilon(\mathfrak{n}_+)$ and $U_\eta^g(\mathfrak{g})$ be the quotient of $U_\varepsilon^g(\mathfrak{g})$ by the ideal generated by elements $z - \eta(z)$, $z \in Z_0 \cap U_\varepsilon^g(\mathfrak{g})$. The algebras $U_\varepsilon^g(\mathfrak{g})$ and $U_\eta^g(\mathfrak{g})$ can be regarded as quantum analogues of the parabolic subalgebras associated to the subset Γ' of simple roots. Let

also $U''_\eta(\mathfrak{g})$ be the subalgebra of $U'_\eta(\mathfrak{g})$ generated by all the elements X_i^\pm and $L_i^{\pm 1}$ such that $\alpha_i \in \Gamma'$. $U''_\eta(\mathfrak{g})$ can be regarded as the semisimple part of the Levi factor $U'_\eta(\mathfrak{g})$.

The following fundamental proposition states that V is in fact induced from a representation of the algebra $U_\eta^g(\mathfrak{g})$.

Proposition 4.4. ([11], Theorem 6.8, [12], §8, Theorem)

(i) The $U_\eta(\mathfrak{g})$ -module V contains a unique irreducible $U_\eta^g(\mathfrak{g})$ -submodule V' which remains irreducible when restricted to $U''_\eta(\mathfrak{g})$.

(ii) The $U_\eta(\mathfrak{g})$ -module V is induced from the $U_\eta^g(\mathfrak{g})$ -module V' ,

$$V = U_\eta(\mathfrak{g}) \otimes_{U_\eta^g(\mathfrak{g})} V',$$

with the left action defined by left multiplication on $U_\eta^g(\mathfrak{g})$. In particular, $\dim V = m^{t/2} \dim V'$, where $t = |\Delta \setminus \Delta'|$.

(iii) The map $V \mapsto V'$ establishes a bijection $\text{Rep}(U_\eta(\mathfrak{g})) \rightarrow \text{Rep}(U''_\eta(\mathfrak{g}))$, and V' can be regarded as an exceptional representation of the algebra $U_\varepsilon(\mathfrak{g}')$, where \mathfrak{g}' is the Lie subalgebra of \mathfrak{g} generated by the Chevalley generators corresponding to $\alpha_i \in \Gamma'$.

5. REALIZATIONS OF QUANTUM GROUPS ASSOCIATED TO WEYL GROUP ELEMENTS

Some important ingredients that will be used in the proof of the main statement in Section 9 are certain subalgebras of the quantum group. These subalgebras are defined in terms of realizations of the algebra $U_\varepsilon(\mathfrak{g})$ associated to Weyl group elements. We introduce these realizations in this section. A similar construction in case of quantum groups $U_q(\mathfrak{g})$ with generic q was introduced in [33].

Let s be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, and \mathfrak{h}' the orthogonal complement, with respect to the Killing form, to the subspace of \mathfrak{h} fixed by the natural action of s on \mathfrak{h} . The restriction of the natural action of s on \mathfrak{h}^* to the subspace \mathfrak{h}'^* has no fixed points. Therefore one can define the Cayley transform $\frac{1+s}{1-s}P_{\mathfrak{h}'^*}$ of the restriction of s to \mathfrak{h}'^* , where $P_{\mathfrak{h}'^*}$ is the orthogonal projection operator onto \mathfrak{h}'^* in \mathfrak{h}^* , with respect to the Killing form.

Recall also that in the classification theory of conjugacy classes in the Weyl group W of the complex simple Lie algebra \mathfrak{g} the so-called primitive (or semi-Coxeter in another terminology) elements play a primary role. The primitive elements $w \in W$ are characterized by the property $\det(1 - w) = \det a$, where a is the Cartan matrix of \mathfrak{g} . According to the results of [8] the element s of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ is primitive in the Weyl group W' of a regular semisimple Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of the form

$$\mathfrak{g}' = \mathfrak{h}' + \sum_{\alpha \in \Delta'} \mathfrak{g}_\alpha,$$

where Δ' is a root subsystem of the root system Δ of \mathfrak{g} , \mathfrak{g}_α is the root subspace of \mathfrak{g} corresponding to root α , and \mathfrak{h}' is a Lie subalgebra of \mathfrak{h} (it coincides with \mathfrak{h}' introduced in Section 5).

Moreover, by Theorem C in [8] s can be represented as a product of two involutions,

$$(5.1) \quad s = s^1 s^2,$$

where $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \dots, \gamma_n$ and $\gamma_{n+1}, \dots, \gamma_{l'}$ are positive and mutually orthogonal, and the roots $\gamma_1, \dots, \gamma_{l'}$ form a linear basis of \mathfrak{h}'^* , in particular l' is the rank of \mathfrak{g}' . Recall that $\gamma_1, \dots, \gamma_{l'}$ form a basis of a subspace $\mathfrak{h}'^* \subset \mathfrak{h}^*$ on which s acts without fixed points. We shall study the matrix elements of the Cayley transform of the restriction of s to \mathfrak{h}'^* with respect the basis $\gamma_1, \dots, \gamma_{l'}$.

Lemma 5.1. ([33], Lemma 6.2) *Let $P_{\mathfrak{h}'^*}$ be the orthogonal projection operator onto \mathfrak{h}'^* in \mathfrak{h}^* , with respect to the Killing form. Then the matrix elements of the operator $\frac{1+s}{1-s}P_{\mathfrak{h}'^*}$ in the basis $\gamma_1, \dots, \gamma_{l'}$*

are of the form:

$$(5.2) \quad \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \gamma_i, \gamma_j \right) = \varepsilon_{ij}(\gamma_i, \gamma_j),$$

where

$$\varepsilon_{ij} = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases}.$$

Let γ_i^* , $i = 1, \dots, l'$ be the basis of \mathfrak{h}'^* dual to γ_i , $i = 1, \dots, l'$ with respect to the restriction of the bilinear form (\cdot, \cdot) to \mathfrak{h}'^* . Since the numbers (γ_i, γ_j) are integer each element γ_i^* has the form $\gamma_i^* = \sum_{j=1}^{l'} m_{ij} \gamma_j$, where $m_{ij} \in \mathbb{Q}$. Therefore by the previous lemma the numbers

$$(5.3) \quad \begin{aligned} p_{ij} &= \frac{1}{d_j} \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i, \alpha_j \right) = \\ &= \frac{1}{d_j} \sum_{k,l,p,q=1}^{l'} \gamma_k(\alpha_i) \gamma_l(\alpha_j) \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \gamma_p, \gamma_q \right) m_{kp} m_{lq}, \quad i, j = 1, \dots, l \end{aligned}$$

are rational. Let d be a positive integer such that $p_{ij} \in \frac{1}{d} \mathbb{Z}$ for any $i < j$ (or $i > j$), $i, j = 1, \dots, l$.

Now we suggest a new realization of the quantum group $U_\varepsilon(\mathfrak{g})$ associated to $s \in W$. Fix a positive integer number $n \in \mathbb{N}$, $n > 0$. Assume that $\varepsilon^{2d_i} \neq 1$. Let $U_\varepsilon^s(\mathfrak{g})$ be the associative algebra over \mathbb{C} generated by elements $e_i, f_i, L_i^{\pm 1}$, $i = 1, \dots, l$ subject to the relations:

$$(5.4) \quad \begin{aligned} [L_i, L_j] &= 0, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_i e_j L_i^{-1} = \varepsilon_i^{\delta_{ij}} e_j, \quad L_i f_j L_i^{-1} = \varepsilon_i^{-\delta_{ij}} f_j, \quad \varepsilon_i = \varepsilon^{d_i}, \\ e_i f_j - \varepsilon^{c_{ij}} f_j e_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{\varepsilon_i - \varepsilon_i^{-1}}, \quad c_{ij} = nd \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i, \alpha_j \right), \\ \text{where } K_i &= \prod_{j=1}^l L_j^{a_{ji}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \varepsilon^{rc_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{\varepsilon_i} (e_i)^{1-a_{ij}-r} e_j (e_i)^r &= 0, \quad i \neq j, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \varepsilon^{rc_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{\varepsilon_i} (f_i)^{1-a_{ij}-r} f_j (f_i)^r &= 0, \quad i \neq j. \end{aligned}$$

Theorem 5.2. Assume that $\varepsilon^{2d_i} \neq 1$. For every solution $n_{ij} \in \mathbb{Z}$, $i, j = 1, \dots, l$ of equations

$$(5.5) \quad d_j n_{ij} - d_i n_{ji} = c_{ij}$$

there exists an algebra isomorphism $\psi_{\{n\}} : U_\varepsilon^s(\mathfrak{g}) \rightarrow U_\varepsilon(\mathfrak{g})$ defined by the formulas:

$$\begin{aligned} \psi_{\{n\}}(e_i) &= X_i^+ \prod_{p=1}^l L_p^{n_{ip}}, \\ \psi_{\{n\}}(f_i) &= \prod_{p=1}^l L_p^{-n_{ip}} X_i^-, \\ \psi_{\{n\}}(L_i^{\pm 1}) &= L_i^{\pm 1}. \end{aligned}$$

The proof of this theorem is similar to the proof of Theorem 4.1 in [31].

Remark 5.2. The general solution of equation (5.5) is given by

$$(5.6) \quad n_{ij} = \frac{1}{2d_j} (c_{ij} + s_{ij}),$$

where $s_{ij} = s_{ji}$. If $p_{ij} \in \frac{1}{d}\mathbb{Z}$ for any $i < j$, we put

$$s_{ij} = \begin{cases} c_{ij} & i < j \\ 0 & i = j \\ -c_{ij} & i > j \end{cases}.$$

Then

$$n_{ij} = \begin{cases} \frac{1}{d_j}c_{ij} & i < j \\ 0 & i = j \\ 0 & i > j \end{cases}.$$

By the choice of c_{ij} and d we have $\frac{1}{d_j}c_{ij} = \frac{nd}{d_j} \left(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \alpha_i, \alpha_j \right) = ndp_{ij} \in n\mathbb{Z}$ for $i < j$, $i, j = 1, \dots, l$. Therefore $n_{ij} \in \mathbb{Z}$ for any $i, j = 1, \dots, l$, and integer valued solutions to equations (5.5) exist if $p_{ij} \in \frac{1}{d}\mathbb{Z}$ for any $i < j$. A similar consideration shows that if $p_{ij} \in \frac{1}{d}\mathbb{Z}$ for any $i > j$ integer valued solutions to equations (5.5) exist as well.

We call the algebra $U_\varepsilon^s(\mathfrak{g})$ the realization of the quantum group $U_\varepsilon(\mathfrak{g})$ corresponding to the element $s \in W$.

Remark 5.3. Let $n_{ij} \in \mathbb{Z}$ be a solution of the homogeneous system that corresponds to (5.5),

$$(5.7) \quad d_i n_{ji} - d_j n_{ij} = 0.$$

Then the map defined by

$$(5.8) \quad \begin{aligned} X_i^+ &\mapsto X_i^+ \prod_{p=1}^l L_p^{n_{ip}}, \\ X_i^- &\mapsto \prod_{p=1}^l L_p^{-n_{ip}} X_i^-, \\ L_i^{\pm 1} &\mapsto L_i^{\pm 1} \end{aligned}$$

is an automorphism of $U_\varepsilon(\mathfrak{g})$. Therefore for given element $s \in W$ the isomorphism $\psi_{\{n\}}$ is defined uniquely up to automorphisms (5.8) of $U_\varepsilon(\mathfrak{g})$.

Now we shall study the algebraic structure of $U_\varepsilon^s(\mathfrak{g})$. Denote by $U_\varepsilon^s(\mathfrak{n}_\pm)$ the subalgebra in $U_\varepsilon^s(\mathfrak{g})$ generated by $e_i (f_i), i = 1, \dots, l$. Let $U_\varepsilon^s(\mathfrak{h})$ be the subalgebra in $U_\varepsilon^s(\mathfrak{g})$ generated by $L_i^{\pm 1}, i = 1, \dots, l$.

We shall construct a Poincaré–Birkhoff–Witt basis for $U_\varepsilon^s(\mathfrak{g})$.

Proposition 5.3. (i) For any integer valued solution of equation (5.5) and any normal ordering of the root system Δ_+ the elements $e_\beta = \psi_{\{n\}}^{-1} (X_\beta^+ \prod_{i,j=1}^l L_j^{c_i n_{ij}})$ and $f_\beta = \psi_{\{n\}}^{-1} (\prod_{i,j=1}^l L_j^{-c_i n_{ij}} X_\beta^-)$, $\beta = \sum_{i=1}^l c_i \alpha_i \in \Delta_+$ lie in the subalgebras $U_\varepsilon^s(\mathfrak{n}_+)$ and $U_\varepsilon^s(\mathfrak{n}_-)$, respectively. The elements f_β , $\beta \in \Delta_+$ satisfy the following commutation relations

$$(5.9) \quad f_\alpha f_\beta - \varepsilon^{(\alpha, \beta) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \alpha, \beta)} f_\beta f_\alpha = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C'(k_1, \dots, k_n) f_{\delta_n}^{k_n} f_{\delta_{n-1}}^{k_{n-1}} \dots f_{\delta_1}^{k_1}, \quad \alpha < \beta,$$

where $C'(k_1, \dots, k_n) \in \mathbb{C}$.

(ii) Moreover, the elements $(e)^\mathbf{r} = (e_{\beta_1})^{r_1} \dots (e_{\beta_D})^{r_D}$, $(f)^\mathbf{t} = (f_{\beta_D})^{t_D} \dots (f_{\beta_1})^{t_1}$ and $L^\mathbf{s} = L_1^{s_1} \dots L_l^{s_l}$ for $\mathbf{r}, \mathbf{t} \in \mathbb{N}^l$, $\mathbf{s} \in \mathbb{Z}^l$ form bases of $U_\varepsilon^s(\mathfrak{n}_+)$, $U_\varepsilon^s(\mathfrak{n}_-)$ and $U_\varepsilon^s(\mathfrak{h})$, and the products $(f)^\mathbf{t} L^\mathbf{s} (e)^\mathbf{r}$ form a basis of $U_\varepsilon^s(\mathfrak{g})$. In particular, multiplication defines an isomorphism of vector spaces,

$$U_\varepsilon^s(\mathfrak{n}_-) \otimes U_\varepsilon^s(\mathfrak{h}) \otimes U_\varepsilon^s(\mathfrak{n}_+) \rightarrow U_\varepsilon^s(\mathfrak{g}).$$

(iii) The subalgebra $Z_0 \subset U_\varepsilon^s(\mathfrak{g})$ is the polynomial algebra with generators e_α^m, f_α^m , $\alpha \in \Delta_+$ and $L_i^{\pm 1}$, $i = 1, \dots, l$.

(iv) $U_\varepsilon^s(\mathfrak{g})$ is a free Z_0 -module with basis the set of monomials $(f)^r L^s(e)^t$ for which $0 \leq r_k, t_k, s_i < m$ for $i = 1, \dots, l$, $k = 1, \dots, D$.

The proof of this proposition is similar to the proof of Proposition 4.2 in [33].

6. NILPOTENT SUBALGEBRAS AND QUANTUM GROUPS

In this section we define the subalgebras of $U_\varepsilon(\mathfrak{g})$ which resemble nilpotent subalgebras in \mathfrak{g} and possess nontrivial characters. We start by recalling the definition of certain normal orderings of root systems associated to Weyl group elements (see [33], Section 5 for more details). The definition of subalgebras of $U_\varepsilon(\mathfrak{g})$ having nontrivial characters will be given in terms of root vectors associated to such normal orderings.

Let s , as in the previous section, be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{h}_\mathbb{R}$ the real form of \mathfrak{h} , the real linear span of simple coroots in \mathfrak{h} . The set of roots Δ is a subset of the dual space $\mathfrak{h}_\mathbb{R}^*$.

The Weyl group element s naturally acts on $\mathfrak{h}_\mathbb{R}$ as an orthogonal transformation with respect to the scalar product induced by the Killing form of \mathfrak{g} . Using the spectral theory of orthogonal transformations we can decompose $\mathfrak{h}_\mathbb{R}$ into a direct orthogonal sum of s -invariant subspaces,

$$(6.1) \quad \mathfrak{h}_\mathbb{R} = \bigoplus_{i=0}^K \mathfrak{h}_i,$$

where we assume that \mathfrak{h}_0 is the linear subspace of $\mathfrak{h}_\mathbb{R}$ fixed by the action of s , and each of the other subspaces $\mathfrak{h}_i \subset \mathfrak{h}_\mathbb{R}$, $i = 1, \dots, K$, is either two-dimensional or one-dimensional and the Weyl group element s acts on it as rotation with angle θ_i , $0 < \theta_i \leq \pi$ or as reflection with respect to the origin (which also can be regarded as rotation with angle π). Note that since s has finite order $\theta_i = \frac{2\pi}{m_i}$, $m_i \in \mathbb{N}$.

Since the number of roots in the root system Δ is finite one can always choose elements $h_i \in \mathfrak{h}_i$, $i = 0, \dots, K$, such that $h_i(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s -invariant subspace \mathfrak{h}_i with respect to the natural pairing between $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$.

Now we consider certain s -invariant subsets of roots $\overline{\Delta}_i$, $i = 0, \dots, K$, defined as follows

$$(6.2) \quad \overline{\Delta}_i = \{\alpha \in \Delta : h_j(\alpha) = 0, j > i, h_i(\alpha) \neq 0\},$$

where we formally assume that $h_{K+1} = 0$. Note that for some indexes i the subsets $\overline{\Delta}_i$ are empty, and that the definition of these subsets depends on the order of terms in direct sum (6.1).

Now consider the nonempty s -invariant subsets of roots $\overline{\Delta}_{i_k}$, $k = 0, \dots, M$. For convenience we assume that indexes i_k are labeled in such a way that $i_j < i_k$ if and only if $j < k$. According to this definition $\overline{\Delta}_{i_0} = \{\alpha \in \Delta : s\alpha = \alpha\}$ is the set of roots fixed by the action of s . Observe also that the root system Δ is the disjoint union of the subsets $\overline{\Delta}_{i_k}$,

$$\Delta = \bigcup_{k=0}^M \overline{\Delta}_{i_k}.$$

Now assume that

$$(6.3) \quad |h_{i_k}(\alpha)| > \left| \sum_{l \leq j < k} h_{i_j}(\alpha) \right|, \text{ for any } \alpha \in \overline{\Delta}_{i_k}, \quad k = 0, \dots, M, \quad l < k.$$

Condition (6.3) can be always fulfilled by suitable rescalings of the elements h_{i_k} .

Consider the element

$$(6.4) \quad \bar{h} = \sum_{k=0}^M h_{i_k} \in \mathfrak{h}_\mathbb{R}.$$

From definition (6.2) of the sets $\overline{\Delta}_i$ we obtain that for $\alpha \in \overline{\Delta}_{i_k}$

$$(6.5) \quad \bar{h}(\alpha) = \sum_{j \leq k} h_{i_j}(\alpha) = h_{i_k}(\alpha) + \sum_{j < k} h_{i_j}(\alpha)$$

Now condition (6.3), the previous identity and the inequality $|x + y| \geq ||x| - |y||$ imply that for $\alpha \in \overline{\Delta}_{i_k}$ we have

$$|\bar{h}(\alpha)| \geq ||h_{i_k}(\alpha)| - |\sum_{j < k} h_{i_j}(\alpha)|| > 0.$$

Since Δ is the disjoint union of the subsets $\overline{\Delta}_{i_k}$, $\Delta = \bigcup_{k=0}^M \overline{\Delta}_{i_k}$, the last inequality ensures that \bar{h} belongs to a Weyl chamber of the root system Δ , and one can define the subset of positive roots Δ_+ and the set of simple positive roots Γ with respect to that chamber. From condition (6.3) and formula (6.5) we also obtain that a root $\alpha \in \overline{\Delta}_{i_k}$ is positive if and only if

$$(6.6) \quad h_{i_k}(\alpha) > 0.$$

We denote by $(\overline{\Delta}_{i_k})_+$ the set of positive roots contained in $\overline{\Delta}_{i_k}$, $(\overline{\Delta}_{i_k})_+ = \Delta_+ \cap \overline{\Delta}_{i_k}$.

We shall also need a parabolic subalgebra \mathfrak{p} of \mathfrak{g} associated to the semisimple element $\bar{h}_0 = \sum_{k=1}^M h_{i_k} \in \mathfrak{h}_{\mathbb{R}}$. This subalgebra is defined with the help of the linear eigenspace decomposition of \mathfrak{g} with respect to the adjoint action of \bar{h}_0 on \mathfrak{g} , $\mathfrak{g} = \bigoplus_m (\mathfrak{g})_m$, $(\mathfrak{g})_m = \{x \in \mathfrak{g} \mid [\bar{h}_0, x] = mx\}$, $m \in \mathbb{R}$. By definition $\mathfrak{p} = \bigoplus_{m \geq 0} (\mathfrak{g})_m$ is a parabolic subalgebra in \mathfrak{g} , $\mathfrak{n} = \bigoplus_{m > 0} (\mathfrak{g})_m$ and $\mathfrak{l} = \{x \in \mathfrak{g} \mid [\bar{h}_0, x] = 0\}$ are the nilradical and the Levi factor of \mathfrak{p} , respectively. Note that we have natural inclusions of Lie algebras $\mathfrak{p} \supset \mathfrak{b}_+ \supset \mathfrak{n}$, where \mathfrak{b}_+ is the Borel subalgebra of \mathfrak{g} corresponding to the system Γ of simple roots, and Δ_{i_0} is the root system of the reductive Lie algebra \mathfrak{l} . We also denote by $\bar{\mathfrak{n}}$ the nilpotent subalgebra opposite to \mathfrak{n} , $\bar{\mathfrak{n}} = \bigoplus_{m < 0} (\mathfrak{g})_m$.

For every element $w \in W$ one can introduce the set $\Delta_w = \{\alpha \in \Delta_+ : w(\alpha) \in -\Delta_+\}$, and the number of the elements in the set Δ_w is equal to the length $l(w)$ of the element w with respect to the system Γ of simple roots in Δ_+ .

Now recall that s can be represented as a product of two involutions,

$$(6.7) \quad s = s^1 s^2,$$

where $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \dots, \gamma_n$ and $\gamma_{n+1}, \dots, \gamma_{l'}$ are positive and mutually orthogonal, and the roots $\gamma_1, \dots, \gamma_{l'}$ form a linear basis of \mathfrak{h}' , in particular l' is the rank of a regular subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ (see formula (5.1)).

Proposition 6.1. ([33], **Proposition 5.1**) *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ the system of positive roots defined with the help of element (6.4), $\Delta_+ = \{\alpha \in \Delta \mid \bar{h}(\alpha) > 0\}$.*

Then there is a normal ordering of the root system Δ_+ of the following form

$$(6.8) \quad \begin{aligned} & \beta_1^1, \dots, \beta_t^1, \beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \\ & \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1, \dots, \\ & \beta_1^2, \dots, \beta_q^2, \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \\ & \gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2, \beta_{2q+2m_{l(s^2)}-(l'-n)+1}^2, \dots, \beta_{l(s^2)}^2, \\ & \beta_1^0, \dots, \beta_{D_0}^0, \end{aligned}$$

where

$$\begin{aligned} & \{\beta_1^1, \dots, \beta_t^1, \beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \\ & \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1\} = \Delta_{s^1}, \end{aligned}$$

$$\begin{aligned}
 & \{\beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \\
 & \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n\} = \{\alpha \in \Delta_+ | s^1(\alpha) = -\alpha\}, \\
 & \{\beta_1^2, \dots, \beta_q^2, \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \\
 & \gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2, \beta_{2q+2m_{l(s^2)}-(l'-n)+1}^2, \dots, \beta_{l(s^2)}^2\} = \Delta_{s^2}, \\
 & \{\gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \\
 & \gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2\} = \{\alpha \in \Delta_+ | s^2(\alpha) = -\alpha\}, \\
 & \{\beta_1^0, \dots, \beta_{D_0}^0\} = \Delta_0 = \{\alpha \in \Delta_+ | s(\alpha) = \alpha\},
 \end{aligned}$$

and s^1, s^2 are the involutions entering decomposition (5.1), $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \dots, \gamma_n$ and $\gamma_{n+1}, \dots, \gamma_{l'}$ are positive and mutually orthogonal.

The length of the ordered segment $\Delta_{\mathbf{m}_+} \subset \Delta$ in normal ordering (6.8),

$$\begin{aligned}
 \Delta_{\mathbf{m}_+} = & \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \\
 & \gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1, \dots, \beta_1^2, \dots, \beta_q^2, \\
 & \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \gamma_{l'},
 \end{aligned} \tag{6.9}$$

is equal to

$$D - \left(\frac{l(s) - l'}{2} + D_0 \right), \tag{6.10}$$

where D is the number of roots in Δ_+ , $l(s)$ is the length of s and D_0 is the number of positive roots fixed by the action of s .

Moreover, for any two roots $\alpha, \beta \in \Delta_{\mathbf{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ can not be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \dots < \gamma_{i_q} < \beta$.

Now we can define the subalgebras of $U_\varepsilon(\mathfrak{g})$ which resemble nilpotent subalgebras in \mathfrak{g} and possess nontrivial characters.

Theorem 6.2. Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. Fix a decomposition (5.1) of s and let Δ_+ be the system of positive roots associated to s . Assume that $\varepsilon^{2d_i} \neq 1$ and that $\varepsilon^{nd-1} = 1$, where d and n are defined in Section 5. Let $U_\varepsilon^s(\mathfrak{g})$ be the realization of the quantum group $U_\varepsilon(\mathfrak{g})$ associated to s . Let $f_\beta \in U_\varepsilon^s(\mathfrak{n}_-)$, $\beta \in \Delta_+$ be the root vectors associated to the corresponding normal ordering (6.8) of Δ_+ .

Then elements $f_\beta \in U_\varepsilon^s(\mathfrak{n}_-)$, $\beta \in \Delta_{\mathbf{m}_+}$, where $\Delta_{\mathbf{m}_+} \subset \Delta$ is ordered segment (6.9), generate a subalgebra $U_\varepsilon^s(\mathfrak{m}_-) \subset U_\varepsilon^s(\mathfrak{g})$. The elements $f^{\mathbf{r}} = f_{\beta_D}^{r_D} \dots f_{\beta_1}^{r_1}$, $r_i \in \mathbb{N}$, $i = 1, \dots, D$ and r_i can be strictly positive only if $\beta_i \in \Delta_{\mathbf{m}_+}$, form a linear basis of $U_\varepsilon^s(\mathfrak{m}_-)$.

Moreover the map $\chi^s : U_\varepsilon^s(\mathfrak{m}_-) \rightarrow \mathbb{C}$ defined on generators by

$$\chi^s(f_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \dots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i, c_i \in \mathbb{C} \end{cases} \tag{6.11}$$

is a character of $U_\varepsilon^s(\mathfrak{m}_-)$.

Proof. The first statement of the theorem follows straightforwardly from commutation relations (5.9) and Proposition 5.3.

In order to prove that the map $\chi^s : U_\varepsilon^s(\mathfrak{m}_-) \rightarrow \mathbb{C}$ defined by (6.11) is a character of $U_\varepsilon^s(\mathfrak{m}_-)$ we show that all relations (5.9) for f_α, f_β with $\alpha, \beta \in \Delta_{\mathbf{m}_+}$, which are obviously defining relations in the subalgebra $U_\varepsilon^s(\mathfrak{m}_-)$, belong to the kernel of χ^s . By definition the only generators of $U_\varepsilon^s(\mathfrak{m}_-)$ on which χ^s does not vanish are f_{γ_i} , $i = 1, \dots, l'$. By the last statement in Proposition 6.1 for any two

roots $\alpha, \beta \in \Delta_{\mathfrak{m}+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ can not be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \dots < \gamma_{i_q} < \beta$. Hence for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}+}$ such that $\alpha < \beta$ the value of the map χ^s on the l.h.s. of the corresponding commutation relation (5.9) is equal to zero.

Therefore it suffices to prove that

$$\chi^s(f_{\gamma_i} f_{\gamma_j} - \varepsilon^{(\gamma_i, \gamma_j) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_i, \gamma_j)} f_{\gamma_j} f_{\gamma_i}) = c_i c_j (1 - \varepsilon^{(\gamma_i, \gamma_j) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_i, \gamma_j)}) = 0, \quad i < j.$$

Since $\varepsilon^{nd-1} = 1$ and $(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_i, \gamma_j)$ are integer numbers for any $i, j = 1, \dots, l'$, the last identity holds provided $(\gamma_i, \gamma_j) + (\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_i, \gamma_j) = 0$ for $i < j$. As we saw in the Lemma 5.1 this is indeed the case. This completes the proof. \square

7. SOME FACTS ABOUT THE GEOMETRY OF THE CONJUGATION ACTION

In this section we collect some results on the geometry of the conjugation action that will be used later. Let $r \in \text{End } \mathfrak{g}$ be a linear operator on \mathfrak{g} satisfying the classical modified Yang–Baxter equation,

$$[rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y], \quad X, Y \in \mathfrak{g}.$$

One can check that if we define operators $r_{\pm} \in \text{End } \mathfrak{g}$ by

$$r_{\pm} = \frac{1}{2}(r \pm id)$$

then the linear subspace $\mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}$, $\mathfrak{g}^* = \{(X_+, X_-), X_{\pm} = r_{\pm} X, X \in \mathfrak{g}\}$ is a Lie subalgebra in $\mathfrak{g} \oplus \mathfrak{g}$ (see, for instance, [30]). We denote by G^* the corresponding subgroup in $G \times G$.

Let $r^0, r^s \in \text{End } \mathfrak{g}$ be the linear operators on \mathfrak{g} defined by

$$r^0 = P_+ - P_-, \quad r^s = P_- - P_+ + \frac{1+s}{1-s} P_{\mathfrak{h}'},$$

where P_+, P_- and $P_{\mathfrak{h}'}$ are the projection operators onto $\mathfrak{n}_+, \mathfrak{n}_-$ and \mathfrak{h}' in the direct sum

$$(7.1) \quad \mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h}' + \mathfrak{h}'^{\perp} + \mathfrak{n}_-,$$

and \mathfrak{h}'^{\perp} is the orthogonal complement to \mathfrak{h}' in \mathfrak{h} with respect to the Killing form. One can check that both r^0 and r^s satisfy the classical modified Yang–Baxter equation. Therefore one can define the corresponding subgroups $G_0^*, G_s^* \subset G \times G$

Note also that

$$r_+^s = P_+ + \frac{1}{1-s} P_{\mathfrak{h}'} + \frac{1}{2} P_{\mathfrak{h}'^{\perp}}, \quad r_-^s = -P_- + \frac{s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2} P_{\mathfrak{h}'^{\perp}},$$

where $P_{\mathfrak{h}'^{\perp}}$ is the projection operator onto \mathfrak{h}'^{\perp} in direct sum (7.1). Hence every element $(L_+, L_-) \in G_s^*$ may be uniquely written as

$$(7.2) \quad (L_+, L_-) = (h_+, h_-)(n_+, n_-),$$

where $n_{\pm} \in N_{\pm}$, $h_+ = \exp((\frac{1}{1-s} P_{\mathfrak{h}'} + \frac{1}{2} P_{\mathfrak{h}'^{\perp}})x)$, $h_- = \exp((\frac{s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2} P_{\mathfrak{h}'^{\perp}})x)$, $x \in \mathfrak{h}$. In particular, G_s^* is a solvable algebraic subgroup in $G \times G$.

Similarly we have

$$r_+^0 = P_+ + \frac{1}{2} P_{\mathfrak{h}}, \quad r_-^0 = -P_- - \frac{1}{2} P_{\mathfrak{h}}, \quad P_{\mathfrak{h}} = P_{\mathfrak{h}'} + P_{\mathfrak{h}'^{\perp}},$$

and hence every element $(L'_+, L'_-) \in G_0^*$ may be uniquely written as

$$(L'_+, L'_-) = (h'_+, h'_-)(n'_+, n'_-), \quad n_{\pm} \in N_{\pm}, \quad h'_+ = \exp(\frac{1}{2}x'), \quad h'_- = \exp(-\frac{1}{2}x'), \quad x' \in \mathfrak{h}.$$

In particular, G_0^* is also a solvable algebraic subgroup in $G \times G$.

We shall need an isomorphism of varieties $\phi : G_0^* \rightarrow G_s^*$ which is uniquely defined by the requirement that if $\phi(L'_+, L'_-) = (L_+, L_-)$ then

$$(7.3) \quad L = tL't^{-1}, \quad L' = L'_-(L'_+)^{-1}, \quad L = L_-L_+^{-1}, \quad t = e^{Ax'},$$

where $A \in \text{End } \mathfrak{h}$ is the endomorphism of \mathfrak{h} defined by

$$(7.4) \quad AH_i = \frac{1}{2nd} \sum_{j=1}^l \frac{n_{ij}}{d_i} Y_j, \quad i = 1, \dots, l,$$

n_{ij} are solutions to equations (5.5), and

$$Y_i = \sum_{j=1}^l d_i(a^{-1})_{ij} H_j,$$

are the weight-type generators of \mathfrak{h} (see [33] for more detail).

In fact (7.3) is an isomorphism of Poisson manifolds if G_0^* is regarded as the dual Poisson-Lie group to the Poisson-Lie group G equipped with the standard Sklyanin bracket, and G_s^* is regarded as the dual Poisson-Lie group to the Poisson-Lie group G equipped with the Sklyanin bracket associated to the r -matrix r^s (see [33], Section 10). We shall not need this fact in this paper.

Formula (7.2) and decomposition of N_+ into products of one-dimensional subgroups corresponding to roots also imply that every element L_- may be represented in the form

$$(7.5) \quad L_- = \exp \left[\sum_{i=1}^l b_i \left(\frac{s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2} P_{\mathfrak{h}'^\perp} \right) H_i \right] \times \prod_{\beta} \exp[b_{-\beta} X_{-\beta}], \quad b_i, b_{-\beta} \in \mathbb{C},$$

where the product over roots is taken in the same order as in (6.8), and the root vectors $X_{-\beta}$ are constructed as in (3.5) using the normal ordering of Δ_+ opposite to (6.8).

Let M_{\pm} be the subgroups in N_{\pm} corresponding to the Lie subalgebras $\mathfrak{m}_{\pm} \subset \mathfrak{n}_{\pm}$ which are generated by root vectors $X_{\pm\beta}$, $\beta \in \Delta_{\mathfrak{m}_+}$. Now define a map $\mu_{M_+} : G_s^* \rightarrow M_-$ by

$$(7.6) \quad \mu_{M_+}(L_+, L_-) = m_-,$$

where for L_- given by (7.5) m_- is defined as follows

$$(7.7) \quad m_- = \prod_{\beta \in \Delta_{\mathfrak{m}_+}} \exp[b_{-\beta} X_{-\beta}],$$

and the product over roots is taken in the same order as in the normally ordered segment $\Delta_{\mathfrak{m}_+}$.

By definition μ_{M_+} is a morphism of algebraic varieties.

Let u be the element defined by

$$(7.8) \quad u = \prod_{i=1}^{l'} \exp[t_i X_{-\gamma_i}] \in M_-, \quad t_i \in \mathbb{C},$$

where the product over roots is taken in the same order as in the normally ordered segment $\Delta_{\mathfrak{m}_+}$.

Let $X_{\alpha}(t) = \exp(tX_{\alpha}) \in G$, $t \in \mathbb{C}$ be the one-parametric subgroup in the algebraic group G corresponding to root $\alpha \in \Delta$. Recall that for any $\alpha \in \Delta_+$ and any $t \neq 0$ the element $s_{\alpha}(t) = X_{\alpha}(-t)X_{-\alpha}(t)X_{\alpha}(-t) \in G$ is a representative for the reflection s_{α} corresponding to the root α . Denote by $s \in G$ the following representative of the Weyl group element $s \in W$,

$$(7.9) \quad s = s_{\gamma_1}(t_1) \dots s_{\gamma_{l'}}(t_{l'}),$$

where the numbers t_i are defined in (7.8), and we assume that $t_i \neq 0$ for any i .

Let Z be the subgroup of G corresponding to the Lie subalgebra \mathfrak{z} generated by the semisimple part \mathfrak{m} of the Levi subalgebra \mathfrak{l} and by the centralizer of s in \mathfrak{h} . Denote by N the subgroup of G

corresponding to the Lie subalgebra \mathfrak{n} and by \overline{N} the opposite unipotent subgroup in G with the Lie algebra $\overline{\mathfrak{n}} = \bigoplus_{m < 0} (\mathfrak{g})_m$. By definition we have that $N_+ \subset ZN$.

Now assume that the roots $\gamma_1, \dots, \gamma_n$ are simple or the set $\gamma_1, \dots, \gamma_n$ is empty. In that case the complementary subset to $\Delta_{\mathfrak{m}_+}$ in Δ_+ is a segment $\Delta_{\mathfrak{m}_+}^0$ with respect to normal ordering (6.8).

Let $q : G_s^* \rightarrow G$ be the map defined by,

$$q(L_+, L_-) = L_- L_+^{-1}.$$

Consider the space $\mu_{M_+}^{-1}(u)$ which can be explicitly described as follows

$$(7.10) \quad \mu_{M_+}^{-1}(u) = \{(h_+ n_+, s(h_+)ux) | n_+ \in N_+, h_+ \in H, x \in M_-^0\},$$

where M_-^0 is the subgroup of G generated by the one-parametric subgroups corresponding to the roots from the segment $-\Delta_{\mathfrak{m}_+}^0$. Therefore

$$(7.11) \quad q(\mu_{M_+}^{-1}(u)) = \{s(h_+)uxn_+^{-1}h_+^{-1} | n_+ \in N_+, h_+ \in H, x \in M_-^0\}.$$

Proposition 7.1. ([33], Proposition 12.1) *Let $q : G_s^* \rightarrow G$ be the map defined by,*

$$q(L_+, L_-) = L_- L_+^{-1}.$$

Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple or the set $\gamma_1, \dots, \gamma_n$ is empty. Suppose also that the numbers t_i defined in (7.8) are not equal to zero for all i . Then $q(\mu_{M_+}^{-1}(u))$ is a subvariety in $NsZN$ which consists of elements of the form $s(h_+)x''skh_+^{-1}$ with arbitrary $k \in ZN$, $h_+ \in H$ and x'' given by

$$(7.12) \quad x'' = X_{\gamma_1}(u_1)X_{s_{\gamma_1}\gamma_2}(u_2) \dots X_{s_{\gamma_1} \dots s_{\gamma_{l'-1}}\gamma_{l'}}(u_{l'})sx's^{-1} \in N, \quad x' \in \overline{N}, \quad sx's^{-1} \in N,$$

where u_i are arbitrary nonzero complex numbers.

The closure $\overline{q(\mu_{M_+}^{-1}(u))}$ of $q(\mu_{M_+}^{-1}(u))$ is obtained by adding elements of the same form with some u_i equal to 0. The closure $\overline{q(\mu_{M_+}^{-1}(u))}$ is also contained in $NsZN$.

Elements of the form $x''sk$, where $k \in ZN$ and

$$(7.13) \quad x'' = X_{\gamma_1}(t_1)X_{s_{\gamma_1}\gamma_2}(t_2) \dots X_{s_{\gamma_1} \dots s_{\gamma_{l'-1}}\gamma_{l'}}(t_{l'})sx's^{-1} \in N, \quad x' \in \overline{N}, \quad sx's^{-1} \in N$$

can be represented as follows $x''sk = uxn_+^{-1} \in q(\mu_{M_+}^{-1}(u))$, $n_+ \in N_+$, $x \in M_-^0$.

Proposition 7.2. ([32], Propositions 2.1 and 2.2) *Let $N_s = \{v \in N | sv s^{-1} \in \overline{N}\}$. Then the conjugation map*

$$(7.14) \quad N \times sZN_s \rightarrow NsZN$$

is an isomorphism of varieties. Moreover, the variety $\Sigma_s = sZN_s$ is a transversal slice to the set of conjugacy classes in G .

Now we prove a short technical lemma which will play the key role in the proof of the main statement of this paper.

Lemma 7.3. *Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple or the set $\gamma_1, \dots, \gamma_n$ is empty. Suppose also that the numbers t_i defined in (7.8) are not equal to zero for all i . For any $\eta \in NsZN$ and $h \in H$ one can find $n \in N$ such that $n\eta n^{-1} \in q(\mu_{M_+}^{-1}(u))$, and $n\eta n^{-1} = s(h)uxn_+^{-1}h^{-1}$ for some $n_+ \in N_+$, $x \in M_-^0$.*

Proof. Let u_i , $i = 1, \dots, l'$ be nonzero complex numbers such that

$$s(h)X_{\gamma_1}(t_1)X_{s_{\gamma_1}\gamma_2}(t_2) \dots X_{s_{\gamma_1} \dots s_{\gamma_{l'-1}}\gamma_{l'}}(t_{l'})s(h^{-1}) = X_{\gamma_1}(u_1)X_{s_{\gamma_1}\gamma_2}(u_2) \dots X_{s_{\gamma_1} \dots s_{\gamma_{l'-1}}\gamma_{l'}}(u_{l'}).$$

Obviously for any $\eta \in NsZN$ one can find $n \in N$ such that

$$\begin{aligned} n\eta n^{-1} &= X_{\gamma_1}(u_1)X_{s_{\gamma_1}\gamma_2}(u_2)\dots X_{s_{\gamma_1}\dots s_{\gamma_{l'-1}}\gamma_{l'}}(u_{l'})sy = \\ &= s(h)X_{\gamma_1}(t_1)X_{s_{\gamma_1}\gamma_2}(t_2)\dots X_{s_{\gamma_1}\dots s_{\gamma_{l'-1}}\gamma_{l'}}(t_{l'})\tilde{y}h^{-1}, \quad y, \tilde{y} \in ZN. \end{aligned}$$

Now by Proposition 7.1 $n\eta n^{-1}$ can be represented in the form $n\eta n^{-1} = s(h)uxn_+^{-1}h^{-1}$ for some $n_+ \in N_+$, $x \in M_-^0$. This completes the proof. \square

Consider the restriction of the action of G on itself by conjugations to the subgroup M_+ . Denote by $\pi_q : G \rightarrow G/M_+$ the canonical projection onto the quotient with respect to this action.

Proposition 7.4. ([33], **Theorem 12.3**) *Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple or the set $\gamma_1, \dots, \gamma_n$ is empty. Suppose also that the numbers t_i defined in (7.8) are not equal to zero for all i . Then $sZN_s \cap G^0 \subset q(\mu_{M_+}^{-1}(u))$, $sZN_s \subset \overline{q(\mu_{M_+}^{-1}(u))}$, the (locally defined) conjugation action of M_+ on $q(\mu_{M_+}^{-1}(u))$ is (locally) free, the quotient $\pi_q(q(\mu_{M_+}^{-1}(u)))$ is a smooth variety and the algebra of regular functions on $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is isomorphic to the algebra of regular functions on the slice sZN_s .*

Remark 7.4. *Statements similar to Propositions 7.1, 7.4 and Lemma 7.3 can be proved in case when the roots $\gamma_{n+1}, \dots, \gamma_{l'}$ are simple or the set $\gamma_{n+1}, \dots, \gamma_{l'}$ is empty. In that case instead of the map $q : G^* \rightarrow G$ one should use another map $q' : G^* \rightarrow G$, $q'(L_+, L_-) = L_-^{-1}L_+$ which has the same properties as q (see [30], Section 2).*

8. WHITTAKER VECTORS

In this section we introduce the notion of Whittaker vectors for modules over quantum groups at roots of unity and prove an analogue of Engel theorem for them. We start by studying some properties of quantum groups at roots of unity.

From now on we fix an element $s \in W$ and a representation (5.1) for s . We also fix positive integers n and d such that $p_{ij} \in \frac{1}{d}\mathbb{Z}$ for any $i < j$ (or $i > j$), $i, j = 1, \dots, l$, where the numbers p_{ij} are defined by formula (5.3). We shall always assume that $\varepsilon^{2d_i} \neq 1$, $i = 1, \dots, l$ and that $\varepsilon^{nd-1} = 1$. We fix an integer valued solution n_{ij} to equations (5.5) and identify the algebra $U_\varepsilon^{s^{-1}}(\mathfrak{g})$ associated to the Weyl group element s^{-1} with $U_\varepsilon(\mathfrak{g})$ using Theorem 5.2 and the solution $-n_{ij} - \delta_{ij}$ to equations (5.5) (a motivation for adding the extra term $-\delta_{ij}$ to n_{ij} will be given later; as it was explained in Remark 5.3 this term is a solution to homogeneous equations (5.7) and corresponds to an automorphism of $U_\varepsilon(\mathfrak{g})$). Using this identification $U_\varepsilon^{s^{-1}}(\mathfrak{m}_-)$ can be regarded as a subalgebra in $U_\varepsilon(\mathfrak{g})$. Therefore for every character $\eta : Z_0 \rightarrow \mathbb{C}$ one can define the corresponding subalgebra in $U_\eta(\mathfrak{g})$. We denote this subalgebra by $U_\eta(\mathfrak{m}_-)$.

First we study some properties of the finite dimensional algebras $U_\eta(\mathfrak{g})$ and $U_\eta(\mathfrak{m}_-)$. We remind that a finite dimensional algebra is called Frobenius if its left regular representation is isomorphic to the dual of the right regular representation. Thus any free module over a Frobenius algebra is also injective and projective.

Proposition 8.5. *For any character $\eta : Z_0 \rightarrow \mathbb{C}$ the algebra $U_\eta(\mathfrak{g})$ and its subalgebra $U_\eta(\mathfrak{m}_-)$ are Frobenius algebras.*

Proof. The proof of this proposition is parallel to the proof of a similar statement for Lie algebras over fields of prime characteristic (see Proposition 1.2 in [18]). By Theorem 61.3 in [14] it suffices to show that there is a non-degenerate bilinear form $B_\eta : U_\eta(\mathfrak{g}) \times U_\eta(\mathfrak{g}) \rightarrow \mathbb{C}$ which restricts to a non-degenerate bilinear form $B_\eta : U_\eta(\mathfrak{m}_-) \times U_\eta(\mathfrak{m}_-) \rightarrow \mathbb{C}$ and which is associative in the sense that

$$B_\eta(ab, c) = B_\eta(a, bc), \quad a, b, c \in U_\eta(\mathfrak{g}).$$

Consider the free Z_0 -basis of $U_\varepsilon(\mathfrak{g})$ introduced in part (iv) of Proposition 5.3. This basis consists of the monomials $x_I = (f)^{\mathbf{r}} L^{\mathbf{s}}(e)^{\mathbf{t}}$, $I = (r_1, \dots, r_D, s_1, \dots, s_l, t_1, \dots, t_D)$ for which $0 \leq r_k, t_k, s_i < m$ for $i = 1, \dots, l$, $k = 1, \dots, D$. Set $c(x_I) = \sum_{k=1}^D r_k + \sum_{k=1}^D t_k + \sum_{i=1}^l s_i$, $I' = (m-1-r_1, \dots, m-1-r_D, m-1-s_1, \dots, m-1-s_l, m-1-t_1, \dots, m-1-t_D)$ and $P = (m-1, \dots, m-1)$.

Let $\Phi : U_\varepsilon(\mathfrak{g}) \rightarrow Z_0$ be the Z_0 -linear map defined on the basis x_I of monomials by

$$\Phi(x_I) = \begin{cases} 1 & I = P \\ 0 & \text{otherwise} \end{cases}.$$

Using commutation relations (5.4), (5.9) and similar relations for generators f_α one can check that $\Phi(x_I x_J) = 0$ if $c(x_I) + c(x_J) \leq (m-1)(l+2N)$ and $J \neq I'$, while $\Phi(x_I x_{I'}) = c_I \neq 0$. Now by the argument given in the proof of Proposition 1.2 in [18] the discriminant of the associative Z_0 -bilinear pairing $B : U_\varepsilon(\mathfrak{g}) \otimes_{Z_0} U_\varepsilon(\mathfrak{g}) \rightarrow Z_0$, $B(x, y) = \Phi(xy)$ is a unit and the associative bilinear form $B_\eta : U_\eta(\mathfrak{g}) \times U_\eta(\mathfrak{g}) \rightarrow \mathbb{C}$, $B_\eta(x, y) = \eta(B(x, y))$ is non-degenerate. By construction the restriction of B_η , $B_\eta : U_\eta(\mathfrak{m}_-) \times U_\eta(\mathfrak{m}_-) \rightarrow \mathbb{C}$ is non-degenerate and associative as well. This completes the proof. \square

In order to define Whittaker vectors for quantum groups at roots of unity we shall need some auxiliary notions that we are going to discuss now.

Consider the isomorphism of varieties

$$\phi \circ \tilde{\pi} : \text{Spec}(Z_0) \rightarrow G_s^*$$

constructed with the help of the normal ordering of the positive root system Δ_+ opposite to (6.8) and with the help of the solution n_{ij} of equations (5.5). We shall need some property of elements $\eta \in \text{Spec}(Z_0)$ such that $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$. To describe this property we observe that a straightforward calculation using the explicit form of the isomorphism $\psi_{\{-n_{ij}-\delta_{ij}\}}$ shows that the n_- -component \mathbf{Y}^- of the map $\phi \circ \tilde{\pi}$ in the image G_s^* with respect to factorization (7.2) has the form

$$\mathbf{Y}^- : \text{Spec}(Z_0) \rightarrow N_-,$$

$$(8.15) \quad \mathbf{Y}^- = \exp(y_{\beta_D}^- X_{-\beta_D}) \exp(y_{\beta_{D-1}}^- X_{-\beta_{D-1}}) \dots \exp(y_{\beta_1}^- X_{-\beta_1}),$$

where $y_\alpha^- = k_\alpha f_\alpha^m$, for some $k_\alpha \in \mathbb{C}$, $k_\alpha \neq 0$, and y_α^- should be regarded as complex-valued functions on $\text{Spec}(Z_0)$. Note that the elements $f_\alpha \in U_\varepsilon^{(s^{-1})}(\mathfrak{m}_-)$ are constructed using the normal ordering opposite to (6.8), so the order of terms corresponding to roots in the product (8.15) coincides with the order of roots in normal ordering (6.8).

The following property of elements $\eta \in \text{Spec}(Z_0)$, $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$ is a direct consequence of formula (8.15) and of the definition of the variety $\mu_{M_+}^{-1}(u)$ in terms of the map μ_{M_+} (see formulas (7.6), (7.7) and (7.8)).

Lemma 8.6. *Let η be an element of $\text{Spec}(Z_0)$. Assume that $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$. Then for $\beta \in \Delta_{\mathfrak{m}_+}$ we have*

$$(8.16) \quad \eta(f_\beta^m) = \begin{cases} d_i = \frac{t_i}{k_{\gamma_i}} & \beta = \gamma_i, i = 1, \dots, l' \\ 0 & \beta \notin \{\gamma_1, \dots, \gamma_{l'}\} \end{cases}.$$

Finally consider the subalgebra $U_\eta(\mathfrak{h}) \subset U_\eta(\mathfrak{g})$ generated by L_1, \dots, L_l . Since $\eta(L_i^m) \neq 0$, $i = 1, \dots, l$ the elements L_1, \dots, L_l act on any finite-dimensional $U_\eta(\mathfrak{g})$ -module V as mutually commuting semisimple automorphisms. Therefore if by a weight we mean an l -tuple $\omega = (\omega_1, \dots, \omega_l) \in$

$(\mathbb{C}^*)^l$, the space V has a weight decomposition with respect to the action of $U_\eta(\mathfrak{h})$,

$$V = \bigoplus_{\omega \in (\mathbb{C}^*)^l} V_\omega,$$

where

$$V_\omega = \{v \in V, L_i v = \omega_i v, \omega_i \in \mathbb{C}^*, i = 1, \dots, l\}$$

is the weight space corresponding to weight ω .

Observe that by Proposition 6.1 for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ can not be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \dots < \gamma_{i_q} < \beta$, and hence from commutation relations (5.9) one can deduce that the elements $f_\beta \in U_\eta(\mathfrak{m}_-)$, $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \dots, \gamma_{l'}\}$ generate an ideal \mathcal{J} in $U_\eta(\mathfrak{m}_-)$.

Lemma 8.7. *Let η be an element of $\text{Spec}(Z_0)$. Assume that $t_i \neq 0$, $i = 1, \dots, l'$ in formula (7.8) and $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$, so that $\eta(f_{\gamma_i}^m) = d_i \neq 0$, $i = 1, \dots, l'$ and $f_\beta^m = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \dots, \gamma_{l'}\}$. Then the ideal \mathcal{J} is the Jacobson radical of $U_\eta(\mathfrak{m}_-)$ and $U_\eta(\mathfrak{m}_-)/\mathcal{J}$ is isomorphic to the truncated polynomial algebra*

$$\mathbb{C}[f_{\gamma_1}, \dots, f_{\gamma_{l'}}]/\{f_{\gamma_i}^m = d_i\}_{i=1, \dots, l'}.$$

Proof. Since $f_\beta^m = 0$, $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \dots, \gamma_{l'}\}$ we have $\mathcal{J}^m = 0$, and hence the ideal \mathcal{J} is nilpotent. We deduce that \mathcal{J} is contained in the Jacobson radical of $U_\eta(\mathfrak{m}_-)$.

Using commutation relations (5.9) we also have (see the proof of Theorem 6.2)

$$f_{\gamma_i} f_{\gamma_j} - f_{\gamma_j} f_{\gamma_i} \in \mathcal{J}.$$

Therefore the quotient algebra $U_\eta(\mathfrak{m}_-)/\mathcal{J}$ is isomorphic to the truncated polynomial algebra

$$\mathbb{C}[f_{\gamma_1}, \dots, f_{\gamma_{l'}}]/\{f_{\gamma_i}^m = d_i\}_{i=1, \dots, l'}$$

which is semisimple. Therefore \mathcal{J} coincides with the Jacobson radical of $U_\eta(\mathfrak{m}_-)$. \square

Next, commutation relations (5.9) and part (iv) of Proposition 5.3 also imply the following lemma.

Lemma 8.8. *Let β_1, \dots, β_D be the normal ordering of Δ_+ opposite to (6.8). Then for any character $\eta : Z_0 \rightarrow \mathbb{C}$ the elements $f_{\beta_D}^{r_D} \dots f_{\beta_1}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}}$, where $r_i, n_j \in \mathbb{N}$, $0 \leq r_i, n_j \leq m-1$, $i = 1, \dots, D$, $j = 1, \dots, l'$, and r_i can be strictly positive only if $\beta_i \in \Delta_{\mathfrak{m}_+}$, $\beta_i \notin \{\gamma_1, \dots, \gamma_{l'}\}$, form a linear basis of $U_\eta(\mathfrak{m}_-)$.*

The elements $f_{\beta_D}^{r_D} \dots f_{\beta_1}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}}$, $r_i, n_j \in \mathbb{N}$, $0 \leq r_i, n_j \leq m-1$, $i = 1, \dots, D$, $j = 1, \dots, l'$, r_i can be strictly positive only if $\beta_i \in \Delta_{\mathfrak{m}_+}$, $\beta_i \notin \{\gamma_1, \dots, \gamma_{l'}\}$, and at least one r_i is strictly positive, form a linear basis of \mathcal{J} .

In Theorem 6.2 we constructed some characters of the algebra $U_\varepsilon^{s^{-1}}(\mathfrak{m}_-)$. Now we show that the algebra $U_\eta(\mathfrak{m}_-)$ has a unique up to isomorphism irreducible representation which is one-dimensional.

Proposition 8.9. *Let η be an element of $\text{Spec}(Z_0)$. Assume that $t_i \neq 0$, $i = 1, \dots, l'$ in formula (7.8) and $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$, so that $\eta(f_{\gamma_i}^m) = d_i \neq 0$, $i = 1, \dots, l'$. Then all nonzero irreducible representations of the algebra $U_\eta(\mathfrak{m}_-)$ are one-dimensional and have the form*

$$(8.17) \quad \chi(f_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \dots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i \end{cases},$$

where complex numbers c_i satisfy the conditions $c_i^m = d_i$, $i = 1, \dots, l'$. Moreover, all irreducible representations \mathbb{C}_χ of $U_\eta(\mathfrak{m}_-)$ are isomorphic to each other.

Proof. Let V be a nonzero finite-dimensional irreducible $U_\eta(\mathfrak{m}_-)$ -module. By Lemma 8.6 elements of the ideal $\mathcal{J} \subset U_\eta(\mathfrak{m}_-)$ act by nilpotent transformations on V . Therefore from Engel theorem one can deduce that the subspace $V_{\mathcal{J}} = \{v \in V \mid xv = 0 \ \forall x \in \mathcal{J}\}$, $V_{\mathcal{J}} \subset V$, is nonzero.

Using commutation relations (5.9) we have (see the proof of Theorem 6.2)

$$f_{\gamma_i} f_{\gamma_j} - f_{\gamma_j} f_{\gamma_i} \in \mathcal{J}.$$

These relations and the fact that \mathcal{J} is an ideal in $U_\eta(\mathfrak{m}_-)$ imply that the elements $f_{\gamma_1}, \dots, f_{\gamma_{l'}}$ act on $V_{\mathcal{J}}$ by mutually commuting endomorphisms. Note that by Lemma 8.6 $\eta(f_{\gamma_i}^m) = d_i \neq 0$, $i = 1, \dots, l'$ and hence elements f_{γ_i} act on $V_{\mathcal{J}}$ and on V by semisimple automorphisms.

Let $v \in V_{\mathcal{J}}$ be a common eigenvector in $V_{\mathcal{J}}$ for the mutually commuting semisimple automorphisms generated by the action of $f_{\gamma_1}, \dots, f_{\gamma_{l'}}$, $f_{\gamma_i} v = c_i v$, $c_i \neq 0$, $i = 1, \dots, l'$. By construction the one-dimensional subspace generated by v in V is a submodule. Since V is irreducible this subspace must coincide with V . Thus V is one-dimensional. If we denote by $\chi : U_\eta(\mathfrak{m}_-) \rightarrow \mathbb{C}$ the character of $U_\eta(\mathfrak{m}_-)$ such that

$$\chi(f_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \dots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i \end{cases}$$

and by \mathbb{C}_χ the corresponding one-dimensional representation of $U_\eta(\mathfrak{m}_-)$ then we have $V = \mathbb{C}_\chi$.

Now we have to prove that the representations \mathbb{C}_χ are isomorphic for different characters χ . Note that $\eta(f_{\gamma_i}^m) = d_i \neq 0$, $i = 1, \dots, l'$ and hence we have the following relations in $U_\eta(\mathfrak{m}_-)$: $f_{\gamma_i}^m = d_i$, $i = 1, \dots, l'$. These relations imply that $\chi(f_{\gamma_i}^m) = c_i^m = d_i$, $i = 1, \dots, l'$. Therefore for given η such that $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$ there are only finitely many possible characters χ .

If χ and χ' are two such characters such that $\chi(f_{\gamma_i}) = c_i$, $i = 1, \dots, l'$ and $\chi'(f_{\gamma_i}) = c'_i$, $i = 1, \dots, l'$ then the relations $c_i^m = c'_i{}^m = d_i$, $i = 1, \dots, l'$ imply that $c'_i = \varepsilon^{m_i} c_i$, $0 \leq m_i \leq m - 1$, $m_i \in \mathbb{Z}$, $i = 1, \dots, l'$.

Now observe that for any $h \in \mathfrak{h}$ the map defined by $f_\alpha \mapsto \varepsilon^{\alpha(h)} f_\alpha$, $\alpha \in \Delta_{\mathfrak{m}_+}$ is an automorphism of the algebra $U_\varepsilon^{-1}(\mathfrak{m}_-)$ generated by elements f_α , $\alpha \in \Delta_{\mathfrak{m}_+}$ with defining relations (5.9), and if in addition $\varepsilon^{m\gamma_i(h)} = 1$, $i = 1, \dots, l'$ the above defined map gives rise to an automorphism ς of $U_\eta(\mathfrak{m}_-)$. Indeed in that case $(\varepsilon^{\gamma_i(h)} f_{\gamma_i})^m = f_{\gamma_i}^m$, $i = 1, \dots, l'$ and all the remaining defining relations $\eta(f_{\gamma_i}^m) = d_i \neq 0$, $i = 1, \dots, l'$, $\eta(f_\beta) = 0$, $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \dots, \gamma_{l'}\}$ of the algebra $U_\eta(\mathfrak{m}_-)$ are preserved by the action of the above defined map ς .

Now fix $h \in \mathfrak{h}$ such that $\gamma_i(h) = m_i$, $i = 1, \dots, l'$. Obviously we have $\varepsilon^{mm_i} = 1$, $i = 1, \dots, l'$. We claim that the representation \mathbb{C}_χ twisted by the corresponding automorphism ς coincides with $\mathbb{C}_{\chi'}$. Indeed, we obtain

$$\chi(\varsigma f_{\gamma_i}) = \chi(\varepsilon^{m_i} f_{\gamma_i}) = \varepsilon^{m_i} c_i = c'_i, \quad i = 1, \dots, l'.$$

This establishes the isomorphism $\mathbb{C}_\chi \simeq \mathbb{C}_{\chi'}$ and completes the proof of the proposition. \square

Let V be a $U_\eta(\mathfrak{g})$ -module, η be an element of $\text{Spec}(Z_0)$ such that $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$. Assume that $t_i \neq 0$, $i = 1, \dots, l'$ in formula (7.8). Let $\chi : U_\eta(\mathfrak{m}_-) \rightarrow \mathbb{C}$ be a character defined in the previous proposition, \mathbb{C}_χ the corresponding one-dimensional $U_\eta(\mathfrak{m}_-)$ -module. Then the space $V_\chi = \text{Hom}_{U_\eta(\mathfrak{m}_-)}(\mathbb{C}_\chi, V)$ is called the space of Whittaker vectors of V . Elements of V_χ are called Whittaker vectors.

The following proposition is an analogue of Engel theorem for quantum groups at roots of unity.

Proposition 8.10. *Assume that $t_i \neq 0$, $i = 1, \dots, l'$ in formula (7.8). Suppose also that $Y_j(\sum_{i=1}^{l'} m_i \gamma_i) \neq mp$ for any $m_i \in \{0, \dots, m-1\}$, where at least one of the numbers m_i is nonzero, $p \in \mathbb{Z}$ and $j = 1, \dots, l$. Let η be an element of $\text{Spec}(Z_0)$ such that $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$. Let $\chi : U_\eta(\mathfrak{m}_-) \rightarrow \mathbb{C}$ be a character defined in the previous proposition. Then any nonzero finite-dimensional $U_\eta(\mathfrak{g})$ -module contains a nonzero Whittaker vector.*

Proof. Consider the subalgebra $U_\eta(\mathfrak{m}_- + \mathfrak{h})$ in $U_\eta(\mathfrak{g})$ generated by the elements of $U_\eta(\mathfrak{m}_-)$ and by $L_i^{\pm 1}$, $i = 1, \dots, l$. Let \mathcal{I} be the ideal in $U_\eta(\mathfrak{m}_- + \mathfrak{h})$ generated by \mathcal{J} .

Lemma 8.11. *Assume that $t_i \neq 0$, $i = 1, \dots, l'$ in formula (7.8). Suppose also that $Y_j(\sum_{i=1}^{l'} m_i \gamma_i) \neq mp$ for any $m_i \in \{0, \dots, m-1\}$, where at least one of the numbers m_i is nonzero, $p \in \mathbb{Z}$ and $j = 1, \dots, l$. Let η be an element of $\text{Spec}(Z_0)$ such that $\phi \circ \tilde{\pi}(\eta) \in \mu_{M+}^{-1}(u)$. Let V_0 be a nonzero finite-dimensional $U_\eta(\mathfrak{m}_- + \mathfrak{h})/\mathcal{I}$ -module. Then V_0 is free over the subalgebra \mathcal{A} of $U_\eta(\mathfrak{m}_- + \mathfrak{h})/\mathcal{I}$ generated by the classes of the elements f_{γ_i} , $i = 1, \dots, l'$ in $U_\eta(\mathfrak{m}_- + \mathfrak{h})/\mathcal{I}$, and one can choose a weight \mathcal{A} -basis in V_0 . Fix numbers c_i , $i = 1, \dots, l'$ such that $c_i^m = d_i$, $i = 1, \dots, l'$, where d_i are defined by (8.16). Then the rank of V_0 over \mathcal{A} is equal to the dimension of the subspace of V_0 which consists of elements v such that $f_{\gamma_i} v = c_i v$, $i = 1, \dots, l'$*

Proof. Denote the classes of f_{γ_i} , $i = 1, \dots, l'$ and of $L_i^{\pm 1}$, $i = 1, \dots, l$ in $U_\eta(\mathfrak{m}_- + \mathfrak{h})/\mathcal{I}$ by the same letters. Then $U_\eta(\mathfrak{m}_- + \mathfrak{h})/\mathcal{I}$ has generators f_{γ_i} , $i = 1, \dots, l'$ and $L_i^{\pm 1}$, $i = 1, \dots, l$, and relations

$$L_i L_i^{-1} = 1, \quad L_i L_j = L_j L_i, \quad L_i^m = \eta(L_i), \quad f_{\gamma_i} f_{\gamma_j} = f_{\gamma_j} f_{\gamma_i}, \quad f_{\gamma_i}^m = d_i, \quad L_i f_{\gamma_j} = \varepsilon^{Y_i(\gamma_j)} f_{\gamma_j} L_i.$$

From the relations $L_i^m = \eta(L_i) \neq 0$ and $f_{\gamma_i}^m = d_i$ we obtain that the elements $f_{\gamma_1}, \dots, f_{\gamma_{l'}}$ and L_1, \dots, L_l act on V_0 by semisimple automorphisms. In particular, V_0 has a weight space decomposition for the action of the commutative subalgebra generated by the L_i . If $v \in V_0$ is a vector of weight ω then

$$(8.18) \quad L_j f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} v = \varepsilon^{Y_j(\sum_{i=1}^{l'} n_i \gamma_i)} \omega_j f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} v.$$

Since $Y_j(\sum_{i=1}^{l'} m_i \gamma_i) \neq mp$ for any $m_i \in \{0, \dots, m-1\}$, where at least one of the numbers m_i is nonzero, $p \in \mathbb{Z}$, $j = 1, \dots, l$ and elements f_{γ_i} act on V_0 by semisimple automorphisms, (8.18) implies that the nonzero vectors $f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} v$ have different weights for different l' -tuples $(n_1, \dots, n_{l'})$, $0 \leq n_i \leq m-1$, and hence they are linearly independent in V_0 .

This implies that one can choose linearly independent weight vectors $v_k \in V_0$, $k = 1, \dots, M$ such that

$$V_0 = \bigoplus_{k=1}^M V_0^k \quad (\text{direct sum of } \mathcal{A}\text{-modules}),$$

where V_0^k is the free \mathcal{A} -module with the linear basis $f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} v_k$, for $0 \leq n_i \leq m-1$, $i = 1, \dots, l'$.

One check directly that the vectors

$$\prod_{i=1}^{l'} \sum_{j=0}^{m-1} c_i^{m+1-j} \varepsilon^{-(j+1)p_i} f_{\gamma_i}^j v_k, \quad 0 \leq p_i \leq m-1$$

form another linear basis of V_0^k , and the vector

$$(8.19) \quad w_k = \prod_{i=1}^{l'} \sum_{j=0}^{m-1} c_i^{m+1-j} f_{\gamma_i}^j v_k$$

is the only vector in V_0^k satisfying the conditions $f_{\gamma_i} w_k = c_i w_k$, $i = 1, \dots, l'$. Thus the rank M of V_0 over \mathcal{A} is equal to the dimension of the subspace of such vectors. \square

Now recall that by Lemma 8.6 elements of the ideal $\mathcal{I} \subset U_\eta(\mathfrak{m}_- + \mathfrak{h})$ act by nilpotent transformations on V . Therefore from Engel theorem one can deduce that the subspace $V_{\mathcal{I}} = \{v \in V \mid xv = 0 \ \forall x \in \mathcal{I}\}$, $V_{\mathcal{I}} \subset V$, is nonzero. Now the statement of Proposition 8.10 follows from Lemma 8.11 applied to the $U_\eta(\mathfrak{m}_- + \mathfrak{h})/\mathcal{I}$ -module $V_{\mathcal{I}}$ and the definition of Whittaker vectors. \square

9. SOME PROPERTIES OF FINITE-DIMENSIONAL MODULES OVER QUANTUM GROUPS AT ROOTS OF UNITY

This section is central in the paper. We shall prove that finite-dimensional modules over quantum groups at roots of unity are free over certain subalgebras. More precisely, we have the following theorem.

Theorem 9.12. *Let ζ be an element of $\text{Spec}(Z_0)$. Assume that $Y_j(\sum_{i=1}^{l'} m_i \gamma_i) \neq mp$ for any $m_i \in \{0, \dots, m-1\}$, where at least one m_i is nonzero, $p \in \mathbb{Z}$ and $j = 1, \dots, l$. Assume that the roots $\gamma_1, \dots, \gamma_n$ (or $\gamma_{n+1}, \dots, \gamma_{l'}$) are simple or one of the sets $\gamma_1, \dots, \gamma_n$ or $\gamma_{n+1}, \dots, \gamma_{l'}$ is empty. Suppose also that $t_i \neq 0$, $i = 1, \dots, l'$ in formula (7.8) and that there exists a quantum coadjoint transformation \tilde{g}' such that $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$, where $\eta = \tilde{g}'\zeta$. Then there exists a quantum coadjoint transformation $\tilde{g} \in \mathcal{G}$ such that $\phi \circ \tilde{\pi}(\tilde{g}\eta) \in \mu_{M_+}^{-1}(u)$ and any nonzero finite-dimensional $U_{\tilde{g}\eta}(\mathfrak{g})$ -module V is free over $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ of rank equal to the dimension of the space of Whittaker vectors V_χ , where χ is a character of $U_{\tilde{g}\eta}(\mathfrak{m}_-)$, and hence any nonzero finite-dimensional $U_\zeta(\mathfrak{g})$ -module is free over $\tilde{g}'^{-1}\tilde{g}^{-1}U_{\tilde{g}\eta}(\mathfrak{m}_-)$.*

Proof. Let η' be an element of $\text{Spec}(Z_0)$ satisfying the conditions imposed in the formulation of Theorem 9.12, $\eta = \tilde{g}'\zeta$ and $\tilde{g} \in \mathcal{G}$ an arbitrary quantum coadjoint transformation such that $\phi \circ \tilde{\pi}(\tilde{g}\eta) \in \mu_{M_+}^{-1}(u)$. Let V be a finite-dimensional nonzero $U_{\tilde{g}\eta}(\mathfrak{g})$ -module.

In the proof we shall use the notation of Lemma 8.11. By Lemma 8.6 elements of the ideal $\mathcal{I} \subset U_{\tilde{g}\eta}(\mathfrak{m}_- + \mathfrak{h})$ act by nilpotent transformations on V . Therefore from Engel theorem one can deduce that the subspace $V_{\mathcal{I}} = \{v \in V \mid xv = 0 \ \forall x \in \mathcal{I}\}$, $V_{\mathcal{I}} \subset V$, is nonzero.

By Lemma 8.11 $V_{\mathcal{I}}$ is free over the algebra \mathcal{A} with a weight basis v_k , $k = 1, \dots, M$. As in Lemma 8.11 we denote by $V_{\mathcal{I}}^k$ the free \mathcal{A} -submodule in $V_{\mathcal{I}}$ generated by v_k .

Since the elements f_{γ_i} act on $V_{\mathcal{I}}$ by semisimple automorphisms the m -th powers of which are multiplications by nonzero numbers we can assume that if $V_{\mathcal{I}}^k$ and $V_{\mathcal{I}}^{k'}$ contain vectors of the same weight then the weight of v_k is equal to the weight of $v_{k'}$.

Let $V_{\mathcal{I}}'$ be the linear space with the linear basis $v_k \in V$, $k = 1, \dots, M$. Fix a basis B of V which consists of weight vectors and contains all elements $f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} v_k$, for $0 \leq n_i \leq m-1$, $i = 1, \dots, l'$, $k = 1, \dots, M$. Let $\rho : V \rightarrow V_{\mathcal{I}}'$ be the linear projection such that $\rho v = 0$ for $v \in B$, $v \neq v_k$ for some k . Obviously ρ sends weight vectors to weight vectors.

Consider the left $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module $\text{Hom}_{\mathbb{C}}(U_{\tilde{g}\eta}(\mathfrak{m}_-), V_{\mathcal{I}}')$ with the left $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -action induced by multiplication in $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ from the right. Note that since by Proposition 8.5 the algebra $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ is Frobenius and the space $V_{\mathcal{I}}'$ is finite-dimensional we have a $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module isomorphism $\text{Hom}_{\mathbb{C}}(U_{\tilde{g}\eta}(\mathfrak{m}_-), V_{\mathcal{I}}') \simeq U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V_{\mathcal{I}}'$. Therefore $\text{Hom}_{\mathbb{C}}(U_{\tilde{g}\eta}(\mathfrak{m}_-), V_{\mathcal{I}}')$ is a free $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module.

Now let $\sigma : V \rightarrow \text{Hom}_{\mathbb{C}}(U_{\tilde{g}\eta}(\mathfrak{m}_-), V_{\mathcal{I}}')$ be the homomorphism of $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -modules defined by $\sigma(v)(x) = \rho(xv)$, $x \in U_{\tilde{g}\eta}(\mathfrak{m}_-)$, $v \in V$. We claim that σ is an isomorphism when \tilde{g} is chosen in an appropriate way. Since $\text{Hom}_{\mathbb{C}}(U_{\tilde{g}\eta}(\mathfrak{m}_-), V_{\mathcal{I}}')$ is a free $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module this will imply that V is free over $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ of rank equal to the dimension of $V_{\mathcal{I}}'$. By Lemma 8.11 that dimension is equal to the dimension of the space of Whittaker vectors in V .

First we show that σ is injective. Indeed, the kernel $\text{Ker } \sigma$ of σ is a $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -submodule of V , and hence, by Engel theorem, if $\text{Ker } \sigma \neq \{0\}$ it must contain a nonzero element v annihilated by the nilpotent transformations f_β , $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \dots, \gamma_{l'}\}$. Thus by definition $v \in V_{\mathcal{I}}$. Since $V_{\mathcal{I}}$ is free over \mathcal{A} with basis v_k , v can be uniquely represented as a linear combination of elements $f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} v_k$, for $0 \leq n_i \leq m-1$, $i = 1, \dots, l'$, $k = 1, \dots, M$,

$$v = \sum_{0 \leq n_i \leq m-1, k=1, \dots, M} c_{n_1 \dots n_{l'}, k}^k f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} v_k.$$

Recall that elements f_{γ_i} act on V by automorphisms the m -th powers of which are multiplications by nonzero numbers. Therefore if $c_{n'_1 \dots n'_{l'}}^{k'} \neq 0$ then the element $w = f_{\gamma_{l'}}^{m-n'_{l'}} \dots f_{\gamma_1}^{m-n'_1} v$ can be represented in the form

$$w = \sum_{0 \leq n_i \leq m-1, k=1, \dots, M} d_{n_1 \dots n_{l'}}^k f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} v_k,$$

where $d_{0 \dots 0}^{k'} \neq 0$.

Now we have

$$\sigma(w)(1) = \rho(w) = \sum_{k=1, \dots, M} d_{0 \dots 0}^k v_k,$$

where at least one coefficient $d_{0 \dots 0}^{k'} \neq 0$. Since the elements v_k are linearly independent we deduce that $\sigma(w)(1) \neq 0$, and hence $\sigma(w) \neq 0$.

On the other hand if $v \in \text{Ker } \sigma$ then we also have $w = f_{\gamma_{l'}}^{m-n'_{l'}} \dots f_{\gamma_1}^{m-n'_1} v \in \text{Ker } \sigma$ since $\text{Ker } \sigma$ is a $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -submodule of V . Thus we arrive at a contradiction, and hence σ is injective.

Now we prove that σ is surjective. We start with the following lemma.

Lemma 9.13. *Let η be an element of $\text{Spec}(Z_0)$. Assume that the roots $\gamma_1, \dots, \gamma_n$ (or $\gamma_{n+1}, \dots, \gamma_{l'}$) are simple or one of the sets $\gamma_1, \dots, \gamma_n$ or $\gamma_{n+1}, \dots, \gamma_{l'}$ is empty. Suppose also that $t_i \neq 0$, $i = 1, \dots, l'$ in formula (7.8) and that $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$. Then there exists a quantum coadjoint transformation $\tilde{g} \in \mathcal{G}$ such that $\phi \circ \tilde{\pi}(\tilde{g}\eta) \in \mu_{M_+}^{-1}(u)$ and for any $\alpha \in \Delta_{\mathfrak{m}_+}$, $\alpha \notin \{\gamma_1, \dots, \gamma_{l'}\}$ any nonzero finite-dimensional $U_{\tilde{g}\eta}(\mathfrak{g})$ -module V is free over the subalgebra $U_{\tilde{g}\eta}(f_\alpha)$ of $U_{\tilde{g}\eta}(\mathfrak{g})$ generated by the unit and by f_α .*

Proof. Recall that by Lemma 8.6 for $\alpha \in \Delta_{\mathfrak{m}_+}$, $\alpha \notin \{\gamma_1, \dots, \gamma_{l'}\}$ we have $f_\alpha^m = 0$. Hence $U_{\tilde{g}\eta}(f_\alpha)$ is isomorphic to the truncated polynomial algebra $U_{\tilde{g}\eta}(f_\alpha) = \mathbb{C}[f_\alpha]/\{f_\alpha^m = 0\}$, and in order to show that V is $U_{\tilde{g}\eta}(f_\alpha)$ -free it suffices to verify that all Jordan blocks of the nilpotent endomorphism given by the action of f_α in V have size m .

Denote by V_{f_α} the kernel of f_α in V . Since $f_\alpha^m = 0$ the subspace V_{f_α} is not trivial. Let w_i , $i = 1, \dots, P$ be a linear basis of V_{f_α} . We show that the vectors $e_\alpha^k w_i$, $k = 0, \dots, m-1$, $i = 1, \dots, P$ are linearly independent.

Assume that they are linearly dependant. Then there are nonzero elements $w_k^1 \in V_{f_\alpha}$, $k = 0, \dots, Q$, $Q \leq m-1$ such that

$$(9.20) \quad \sum_{k=0}^Q a_k e_\alpha^k w_k^1 = 0,$$

where $a_k \in \mathbb{C}$ and $a_Q \neq 0$.

Now from formula (13), Sect. 9.3 in [9] we deduce the following commutation relations

$$(9.21) \quad f_\alpha e_\alpha^k = e_\alpha^k f_\alpha - [k]_{\varepsilon_\alpha} e_\alpha^{k-1} \frac{\varepsilon_\alpha^{k-1} K_\alpha - \varepsilon_\alpha^{1-k} K_\alpha^{-1}}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}},$$

where if $\alpha = s_{i_1} \dots s_{i_{p-1}} \alpha_{i_p}$ then $K_\alpha = T_{i_1} \dots T_{i_{p-1}} K_{i_p}$.

Applying f_α to relation (9.20), using commutation relations (9.21) and the fact that $f_\alpha v = 0$ for $v \in V_{f_\alpha}$ we obtain that

$$(9.22) \quad \sum_{k=1}^Q a_k e_\alpha^{k-1} \frac{\varepsilon_\alpha^{k-1} K_\alpha - \varepsilon_\alpha^{1-k} K_\alpha^{-1}}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}} w_k^1 = 0.$$

Now observe that commutation relations $L_i f_\alpha = \varepsilon^{Y_i(\alpha)} f_\alpha L_i$ and the definition of K_α imply that V_{f_α} is an invariant subspace for the action of $K_\alpha^{\pm 1}$. Therefore in (9.22) $\frac{\varepsilon_\alpha^{k-1} K_\alpha - \varepsilon_\alpha^{1-k} K_\alpha^{-1}}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}} w_k^1 \in V_{f_\alpha}$. We

claim that one can choose a quantum coadjoint transformation $\tilde{g} \in \mathcal{G}$ such that $\phi \circ \tilde{\pi}(\tilde{g}\eta) \in \mu_{M_+}^{-1}(u)$ and vectors $\frac{\varepsilon_\alpha^{k-1} K_\alpha - \varepsilon_\alpha^{1-k} K_\alpha^{-1}}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}} w_k^1$ are all nonzero in (9.22).

Let H' be the subgroup of H which corresponds to the Lie subalgebra $\mathfrak{h}' \subset \mathfrak{h}$ and $H'^\perp \subset H$ be the subgroup corresponding to the Lie subalgebra $\mathfrak{h}'^\perp \subset \mathfrak{h}$ so that $H = H'H'^\perp$ (direct product of subgroups). Since any $\alpha \in \Delta_{\mathfrak{m}_+}$, $\alpha \notin \{\gamma_1, \dots, \gamma_\nu\}$ has a nonzero projection onto \mathfrak{h}' one can find $h \in H'$ such that $(h(K_\alpha^m))^{\frac{2}{m}} \neq \varepsilon_\alpha^{2(1-k)}$ for all $\alpha \in \Delta_{\mathfrak{m}_+}$, $\alpha \notin \{\gamma_1, \dots, \gamma_\nu\}$, $k = 1, \dots, m-1$, and for all roots $(h(K_\alpha^m))^{\frac{2}{m}}$ of degree $\frac{2}{m}$ of $h(K_\alpha^m)$. By Lemma 7.3 for $\eta \in \text{Spec}(Z_0)$, $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$ one can find a quantum coadjoint transformation $\tilde{g} \in \mathcal{G}$ such that $\phi \circ \tilde{\pi}(\tilde{g}\eta) \in \mu_{M_+}^{-1}(u)$ and the $H = \text{Spec}(Z_0^0)$ -component $\tilde{g}\eta_0$ of $\tilde{g}\eta$ in $\text{Spec}(Z_0^+) \times \text{Spec}(Z_0^0) \times \text{Spec}(Z_0^-)$ is equal to h . Thus we have $(\tilde{g}\eta(K_\alpha^m))^{\frac{2}{m}} = (\tilde{g}\eta_0(K_\alpha^m))^{\frac{2}{m}} = (h(K_\alpha^m))^{\frac{2}{m}} \neq \varepsilon_\alpha^{2(1-k)}$ for all $\alpha \in \Delta_{\mathfrak{m}_+}$, $\alpha \notin \{\gamma_1, \dots, \gamma_\nu\}$, $k = 1, \dots, m-1$, and for all roots $(\tilde{g}\eta(K_\alpha^m))^{\frac{2}{m}}$ of degree $\frac{2}{m}$ of $\tilde{g}\eta(K_\alpha^m)$.

Since $K_\alpha^m = \tilde{g}\eta(K_\alpha^m)$ in $U_{\tilde{g}\eta}(\mathfrak{g})$ the numbers $(\tilde{g}\eta(K_\alpha^m))^{\frac{2}{m}}$ exhaust all possible eigenvalues of K_α^2 in V , and hence the operators $K_\alpha^2 - \varepsilon_\alpha^{2(1-k)}$ acting in V are invertible for all $\alpha \in \Delta_{\mathfrak{m}_+}$, $\alpha \notin \{\gamma_1, \dots, \gamma_\nu\}$, $k = 1, \dots, m-1$. Therefore the operators

$$\frac{\varepsilon_\alpha^{k-1} K_\alpha - \varepsilon_\alpha^{1-k} K_\alpha^{-1}}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}} = \frac{\varepsilon_\alpha^{k-1} K_\alpha^{-1} (K_\alpha^2 - \varepsilon_\alpha^{2(1-k)})}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}}$$

are invertible as well.

Thus vectors $\frac{\varepsilon_\alpha^{k-1} K_\alpha - \varepsilon_\alpha^{1-k} K_\alpha^{-1}}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}} w_k^1$ are all nonzero in (9.22), and from (9.22) we obtain the following relation

$$\sum_{k=1}^Q a_k e_\alpha^{k-1} w_k^2 = 0,$$

where $w_k^2 \in V_{f_\alpha}$ are nonzero vectors.

Applying successively f_α to the above relation $Q-1$ times and using similar arguments we obtain that

$$a_Q w_k^{Q-1} = 0$$

for a nonzero vector $w_k^{Q-1} \in V_{f_\alpha}$. This is a contradiction. Thus the vectors $e_\alpha^k w_i$, $k = 0, \dots, m-1$, $i = 1, \dots, P$ are linearly independent. The last assertion implies that $\dim V \geq m \dim V_{f_\alpha}$. Since the Jordan blocks of f_α in V have size at most m we also have the opposite inequality, $\dim V \leq m \dim V_{f_\alpha}$. Thus $\dim V = m \dim V_{f_\alpha}$, and hence all Jordan blocks of f_α in V have size m . This completes the proof. \square

From now on we assume that $\tilde{g} \in \mathcal{G}$ is fixed as in the previous lemma. Recall that we already proved that the $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module homomorphism

$$\sigma : V \rightarrow \text{Hom}_{\mathbb{C}}(U_{\tilde{g}\eta}(\mathfrak{m}_-), V'_I) = U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_I$$

is an imbedding. Thus V is a submodule of the free $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module $U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_I$.

Let β_1, \dots, β_D be the normal ordering of Δ_+ opposite to (6.8). Then by Lemma 8.8 the elements $f_{\beta_D}^{r_D} \dots f_{\beta_1}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}}$, where $r_i, n_j \in \mathbb{N}$, $0 \leq r_i, n_j \leq m-1$, $i = 1, \dots, D$, $j = 1, \dots, l'$, and r_i can be strictly positive only if $\beta_i \in \Delta_{\mathfrak{m}_+}$, $\beta_i \notin \{\gamma_1, \dots, \gamma_\nu\}$, form a linear basis of $U_\eta(\mathfrak{m}_-)$. Hence the elements

$$(9.23) \quad f_{\beta_{i_L}}^{r_L} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_k,$$

$\{\beta_{i_1}, \dots, \beta_{i_L}\} = \Delta_{\mathfrak{m}_+} \setminus \{\gamma_1, \dots, \gamma_\nu\}$, $\beta_{i_1} < \dots < \beta_{i_L}$ form a linear basis of $U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_I$.

Lemma 9.14. *There exist elements*

$$(9.24) \quad \bar{v}_k = \sum_{0 \leq n_1, \dots, n_{l'} < m} \sum_{i=1}^M b_{n_1 \dots n_{l'}}^{ik} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i \in U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_L, \quad k = 1, \dots, R, \quad b_{n_1 \dots n_{l'}}^{ik} \in \mathbb{C}$$

such that the elements

$$(9.25) \quad f_{\beta_{i_L}}^{r_L} \dots f_{\beta_{i_1}}^{r_1} \bar{v}_k, \quad \{\beta_{i_1}, \dots, \beta_{i_L}\} = \Delta_{\mathfrak{m}_+} \setminus \{\gamma_1, \dots, \gamma_{l'}\}, \beta_{i_1} < \dots < \beta_{i_L}, \quad 0 \leq r_i \leq m-1, \quad k = 1, \dots, R$$

is a linear basis of the $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -submodule $V \subset U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_L$.

Proof. We prove the statement of the lemma by induction. First, by the previous lemma V is free over $U_{\tilde{g}\eta}(f_{\beta_{i_L}})$. Let v_k^L be a $U_{\tilde{g}\eta}(f_{\beta_{i_L}})$ -basis of V . Consider V as a $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -submodule of $U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_L$. Then using basis (9.23) one can write

$$v_k^L = \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_L < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'} r_1 \dots r_L}^{ik} f_{\beta_{i_L}}^{r_L} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i, \quad c_{n_1 \dots n_{l'} r_1 \dots r_L}^{ik} \in \mathbb{C}.$$

From the last identity we obtain that

$$\begin{aligned} v_k^L - \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_{L-1} < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'} r_1 \dots r_{L-1}, 0}^{ik} f_{\beta_{i_{L-1}}}^{r_{L-1}} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i = \\ = \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_L < m, r_L > 0}} \sum_{i=1}^M c_{n_1 \dots n_{l'} r_1 \dots r_L}^{ik} f_{\beta_{i_L}}^{r_L} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i \in \text{Im } f_{\beta_{i_L}}, \end{aligned}$$

and hence

$$\begin{aligned} v_k^L - \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_{L-1} < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'} r_1 \dots r_{L-1}, 0}^{ik} f_{\beta_{i_{L-1}}}^{r_{L-1}} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i = \\ = \sum_q \sum_{s=1}^{m-1} c_{sq}^k f_{\beta_{i_L}}^s v_q^L, \quad c_{sq}^k \in \mathbb{C}. \end{aligned}$$

Since $f_{\beta_{i_L}}^m = 0$ elements

$$\bar{v}_k^L = v_k^L - \sum_q \sum_{s=1}^{m-1} c_{sq}^k f_{\beta_{i_L}}^s v_q^L = \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_{L-1} < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'} r_1 \dots r_{L-1}, 0}^{ik} f_{\beta_{i_{L-1}}}^{r_{L-1}} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i$$

form another $U_{\tilde{g}\eta}(f_{\beta_{i_L}})$ -basis of V .

Now assume that for given $L > p > 0$ there are elements

$$(9.26) \quad \bar{v}_k^{p+1} = \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_p < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'} r_1 \dots r_p}^{ik} f_{\beta_{i_p}}^{r_p} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i$$

such that

$$(9.27) \quad f_{\beta_{i_L}}^{r_L} \dots f_{\beta_{i_{p+1}}}^{r_{p+1}} \bar{v}_k^{p+1}$$

is a linear basis of V .

The subspace V'_p of V generated by elements (9.26) is $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -invariant since V is $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -invariant, the subspace W_p in $U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_p$ generated by elements of form (9.26) with arbitrary coefficients $c_{n_1 \dots n_{l'}, r_1 \dots r_p}^{ik}$ is $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -invariant and $V'_p = W_p \cap V$.

Using commutation relations (5.9) and the previous observation we immediately obtain that the subspace V_p of V spanned by elements (9.27) with at least one $r_i > 0$ is also $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -invariant. Thus $V = V_p \otimes V'_p$ is a direct sum of $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -modules. Since by the previous lemma V is $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -free its direct summand V'_p is $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -free as well.

Let

$$v_k^p = \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_p < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'}, r_1 \dots r_p}^{ik} f_{\beta_{i_p}}^{r_p} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i$$

be a $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -basis of V'_p . Then we have

$$\begin{aligned} v_k^p - \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_{p-1} < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'}, r_1 \dots r_{p-1} 0}^{ik} f_{\beta_{i_p-1}}^{r_{p-1}} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i = \\ \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_p < m, r_p > 0}} \sum_{i=1}^M c_{n_1 \dots n_{l'}, r_1 \dots r_p}^{ik} f_{\beta_{i_p}}^{r_p} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i \in \text{Im } f_{\beta_{i_p}}, \end{aligned}$$

and hence

$$v_k^p - \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_{p-1} < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'}, r_1 \dots r_{p-1} 0}^{ik} f_{\beta_{i_p-1}}^{r_{p-1}} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i = \sum_q \sum_{s=1}^{m-1} \bar{c}_{sq}^k f_{\beta_{i_p}}^s v_q^p,$$

$\bar{c}_{sq}^k \in \mathbb{C}$.

Since $f_{\beta_{i_p}}^m = 0$ elements

$$\begin{aligned} \bar{v}_k^p = v_k^p - \sum_q \sum_{s=1}^{m-1} \bar{c}_{sq}^k f_{\beta_{i_p}}^s v_q^p = \sum_{\substack{0 \leq n_1, \dots, n_{l'} < m, \\ 0 \leq r_1, \dots, r_{p-1} < m}} \sum_{i=1}^M c_{n_1 \dots n_{l'}, r_1 \dots r_{p-1}}^{ik} f_{\beta_{i_p-1}}^{r_{p-1}} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \otimes v_i, \\ c_{n_1 \dots n_{l'}, r_1 \dots r_{p-1}}^{ik} = c_{n_1 \dots n_{l'}, r_1 \dots r_{p-1} 0}^{ik} \end{aligned}$$

form another $U_{\tilde{g}\eta}(f_{\beta_{i_p}})$ -basis of V'_p , and hence elements

$$f_{\beta_{i_L}}^{r_L} \dots f_{\beta_{i_p}}^{r_p} \bar{v}_k^p$$

form a linear basis of V .

Finally induction over p proves the statement of the lemma. \square

Now consider the linear subspace $\bar{V} \subset V$ generated by elements \bar{v}_k , $k = 1, \dots, R$. By the definition of the elements \bar{v}_k and of the ideal \mathcal{J} the space \bar{V} is isomorphic to the quotient $V/\mathcal{J}V$. Moreover, since elements $L_i^{\pm 1}$ act on V by mutually commuting semisimple automorphisms and normalize the ideal \mathcal{J} in the sense that if $x \in \mathcal{J}$ then $L_i x L_i^{-1} \in \mathcal{J}$, the quotient $V/\mathcal{J}V$ is also isomorphic to the $U_{\tilde{g}\eta}(\mathfrak{m}_- + \mathfrak{h})$ -module $V/\mathcal{I}V$ as a $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module, and $\bar{V} \simeq V/\mathcal{I}V$ as a $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module.

By Lemma 8.11 $V/\mathcal{I}V$ is free over the algebra \mathcal{A} . Obviously, $\mathcal{A} \simeq U_{\tilde{g}\eta}(\mathfrak{m}_-)/\mathcal{J}$, and hence $\bar{V} \simeq V/\mathcal{I}V$ is an isomorphism of \mathcal{A} -modules.

Therefore there are elements $\tilde{v}_k \in \overline{V}$, $k = 1, \dots, T$ such that $f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \tilde{v}_k$ form a linear basis of \overline{V} . Thus by the previous lemma elements

$$f_{\beta_{i_L}}^{r_L} \dots f_{\beta_{i_1}}^{r_1} f_{\gamma_1}^{n_1} \dots f_{\gamma_{l'}}^{n_{l'}} \tilde{v}_k, \quad \{\beta_{i_1}, \dots, \beta_{i_L}\} = \Delta_{\mathfrak{m}_+} \setminus \{\gamma_1, \dots, \gamma_{l'}\},$$

$$\beta_{i_1} < \dots < \beta_{i_L}, \quad 0 \leq r_i, n_i \leq m-1, \quad k = 1, \dots, T$$

form a linear basis of V , and by Lemma 8.8 $V \subset U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_{\mathcal{I}}$ is a free $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -submodule of the free $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module $U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_{\mathcal{I}}$. Therefore V has the form $U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes W$, where $W \subset V'_{\mathcal{I}}$ is a linear subspace. But by construction the subspace $1 \otimes V'_{\mathcal{I}} \subset U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_{\mathcal{I}}$ is contained in the image of the $U_{\tilde{g}\eta}(\mathfrak{m}_-)$ -module homomorphism $\sigma : V \rightarrow U_{\tilde{g}\eta}(\mathfrak{m}_-) \otimes V'_{\mathcal{I}}$. Therefore $W = V'_{\mathcal{I}}$, and σ is surjective. This completes the proof of the theorem. \square

From the previous theorem and the fact that $\dim U_{\tilde{g}\eta}(\mathfrak{m}_-) = m^{\dim \mathfrak{m}_-}$ we immediately obtain the following corollary.

Corollary 9.15. *Assume that the conditions of Theorem 9.12 are satisfied. Then the dimension of any finite-dimensional $U_{\eta}(\mathfrak{g})$ -module V is divisible by $m^{\dim \mathfrak{m}_-}$, and $\dim V = m^{\dim \mathfrak{m}_-} \dim V_{\chi}$.*

Proposition 7.4 implies that $2\dim \mathfrak{m}_- + \dim \Sigma_s = \dim G$. Therefore $\dim \mathfrak{m}_- = \frac{1}{2}(\dim G - \dim \Sigma_s)$. By Proposition 7.2 Σ_s is transversal to the set of conjugacy classes in G . Therefore by Proposition 4.3 and by the definition of maps ϕ and $\tilde{\pi}$ for $\eta \in \text{Spec}(Z_0)$, $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$ we have $\dim G - \dim \Sigma_s \leq \dim \mathcal{O}_{\eta}$, where \mathcal{O}_{η} is the \mathcal{G} -orbit of η . In [11] De Concini, Kac and Procesi formulated the following conjecture.

Conjecture 9.16. (De Concini, Kac and Procesi (1992)) *The dimension of any finite-dimensional irreducible $U_{\eta}(\mathfrak{g})$ -module V is divisible by $m^{\frac{1}{2}\dim \mathcal{O}_{\eta}}$.*

By Proposition 4.4 it suffices to verify this conjecture in case of elements $\eta \in \text{Spec}(Z_0)$ such that $\pi\eta \in G^0$ is exceptional. Recall that by the discussion above for $\eta \in \text{Spec}(Z_0)$, $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$ the dimension of any finite-dimensional $U_{\eta}(\mathfrak{g})$ -module V is divisible by $m^{\frac{1}{2}(\dim G - \dim \Sigma_s)}$. Remind also that the map ϕ is induced by the conjugation action. Combining these facts with the description of the quantum coadjoint action orbits in Proposition 4.3 in terms of the finite covering π we deduce that for $\eta \in \text{Spec}(Z_0)$ such that $\pi\eta \in G^0$ is conjugate to an element from $q\mu_{M_+}^{-1}(u)$ the dimension of any $U_{\eta}(\mathfrak{g})$ -module V is divisible by $m^{\dim \mathfrak{m}_-}$ by Corollary 9.15. Recalling that $\Sigma_s \cap G^0 \subset q(\mu_{M_+}^{-1}(u))$ and $\Sigma_s \subset q(\mu_{M_+}^{-1}(u))$ (see Proposition 7.4), so $\dim \Sigma_s = \dim \Sigma_s \cap G^0$, one obtains that De Concini–Kac–Procesi conjecture follows from the following statement.

Statement *For the conjugacy class \mathcal{O}_g of every exceptional element $g \in G^0$ there exists a Weyl group element $s \in W$ such that the roots $\gamma_1, \dots, \gamma_n$ (or $\gamma_{n+1}, \dots, \gamma_{l'}$) appearing in decomposition (5.1) of s are simple, with respect to a system of positive roots associated to s in Section 6, or one of the sets $\gamma_1, \dots, \gamma_n$ or $\gamma_{n+1}, \dots, \gamma_{l'}$ is empty, $Y_j(\sum_{i=1}^{l'} m_i \gamma_i) \neq mp$ for any $m_i \in \{0, \dots, m-1\}$, where at least one m_i is nonzero, $p \in \mathbb{Z}$ and $j = 1, \dots, l$, and \mathcal{O}_g is strictly transversal to the transversal slice $\Sigma_s \cap G^0$ in the sense that \mathcal{O}_g intersects $\Sigma_s \cap G^0$ and $\dim G - \dim \Sigma_s = \dim \mathcal{O}_g$.*

This statement will be proved in a subsequent paper.

10. A CATEGORIAL EQUIVALENCE

In this section we establish an equivalence between categories of finite-dimensional representations of quantum groups and of q-W algebras at roots of unity. This is a version of Skryabin equivalence for quantum groups at roots of unity (see [29]).

In this section we assume that the conditions of Theorem 9.12 are satisfied. We shall also use the notation introduced in that theorem. For given $\eta \in \text{Spec}(Z_0)$, $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$ we assume that a quantum coadjoint transformation $\tilde{g} \in \mathcal{G}$ is fixed as in Theorem 9.12 and denote $\xi = \tilde{g}\eta \in \text{Spec}(Z_0)$. Let χ be a character of $U_\xi(\mathfrak{m}_-)$, \mathbb{C}_χ the corresponding representation of $U_\xi(\mathfrak{m}_-)$. Denote by Q_χ the induced left $U_\xi(\mathfrak{g})$ -module, $Q_\chi = U_\xi(\mathfrak{g}) \otimes_{U_\xi(\mathfrak{m}_-)} \mathbb{C}_\chi$. Let $W_\varepsilon^s(G) = \text{End}_{U_\xi(\mathfrak{g})}(Q_\chi)^{opp}$ be the algebra of $U_\xi(\mathfrak{g})$ -endomorphisms of Q_χ with the opposite multiplication. The algebra $W_\varepsilon^s(G)$ is called a q-W algebra associated to $s \in W$. Denote by $U_\xi(\mathfrak{g}) - \text{mod}$ the category of finite-dimensional left $U_\xi(\mathfrak{g})$ -modules and by $W_\varepsilon^s(G) - \text{mod}$ the category of finite-dimensional left $W_\varepsilon^s(G)$ -modules. Observe that if $V \in U_\xi(\mathfrak{g}) - \text{mod}$ then the algebra $W_\varepsilon^s(G)$ naturally acts on the finite-dimensional space $V_\chi = \text{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, V) = \text{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, V)$ by compositions of homomorphisms.

Theorem 10.17. *The functor $E \mapsto Q_\chi \otimes_{W_\varepsilon^s(G)} E$ establishes an equivalence of the category of finite-dimensional left $W_\varepsilon^s(G)$ -modules and the category $U_\xi(\mathfrak{g}) - \text{mod}$. The inverse equivalence is given by the functor $V \mapsto V_\chi$. In particular, the latter functor is exact, and every finite-dimensional $U_\xi(\mathfrak{g})$ -module is generated by Whittaker vectors.*

Proof. Let E be a finite-dimensional $W_\varepsilon^s(G)$ -module. First we observe that by the definition of the algebra $W_\varepsilon^s(G)$ we have $W_\varepsilon^s(G) = \text{End}_{U_\xi(\mathfrak{g})}(Q_\chi)^{opp} = \text{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, Q_\chi) = (Q_\chi)_\chi$ as a linear space, and hence $(Q_\chi \otimes_{W_\varepsilon^s(G)} E)_\chi = E$. Therefore to prove the theorem it suffices to check that for any $V \in U_\xi(\mathfrak{g}) - \text{mod}$ the canonical map $f : Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi \rightarrow V$ is an isomorphism.

Indeed, f is injective because otherwise its kernel would contain a nonzero Whittaker vector by Proposition 8.10. But all Whittaker vectors of $Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi$ belong to the subspace $1 \otimes V_\chi$, and the restriction of f to $1 \otimes V_\chi$ induces an isomorphism of the spaces of Whittaker vectors of $Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi$ and of V .

In order to prove that f is surjective we consider the exact sequence

$$0 \rightarrow Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi \rightarrow V \rightarrow W \rightarrow 0,$$

where W is the cokernel of f , and the corresponding long exact sequence of cohomology,

$$\begin{aligned} 0 \rightarrow \text{Ext}_{U_\xi(\mathfrak{m}_-)}^0(\mathbb{C}_\chi, Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi) &\rightarrow \text{Ext}_{U_\xi(\mathfrak{m}_-)}^0(\mathbb{C}_\chi, V) \rightarrow \text{Ext}_{U_\xi(\mathfrak{m}_-)}^0(\mathbb{C}_\chi, W) \rightarrow \\ &\rightarrow \text{Ext}_{U_\xi(\mathfrak{m}_-)}^1(\mathbb{C}_\chi, Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi) \rightarrow \dots \end{aligned}$$

Now recall that f induces an isomorphism of the spaces of Whittaker vectors of $Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi$ and of V . By Theorem 9.12 the finite-dimensional $U_\xi(\mathfrak{g})$ -module $Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi$ is free over $U_\xi(\mathfrak{m}_-)$. Since $U_\xi(\mathfrak{m}_-)$ is Frobenius $Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi$ is also injective over $U_\xi(\mathfrak{m}_-)$, and hence $\text{Ext}_{U_\xi(\mathfrak{m}_-)}^1(\mathbb{C}_\chi, Q_\chi \otimes_{W_\varepsilon^s(G)} V_\chi) = 0$. Therefore the initial part of the long exact cohomology sequence takes the form

$$0 \rightarrow V_\chi \rightarrow V \rightarrow W \rightarrow 0,$$

where the second map in the last sequence is an isomorphism. Using the last exact sequence we deduce that $W_\chi = 0$. But if W is not trivial it will contain a nonzero Whittaker vector by Proposition 8.10. Thus $W = 0$, and f is surjective. This completes the proof of the theorem \square

Next we study some further properties of q-W algebras at roots of unity and of the module Q_χ . First we prove the following lemma.

Lemma 10.18. *The left $U_\xi(\mathfrak{g})$ -module Q_χ is projective in the category $U_\xi(\mathfrak{g}) - \text{mod}$.*

Proof. We have to show that the functor $\text{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, \cdot)$ is exact. Let V^\bullet be an exact complex of finite-dimensional $U_\xi(\mathfrak{g})$ -modules. Since by Theorem 9.12 objects of $U_\xi(\mathfrak{g}) - \text{mod}$ are $U_\xi(\mathfrak{m}_-)$ -free, and $U_\xi(\mathfrak{m}_-)$ is Frobenius we have

$$V^\bullet = U_\xi(\mathfrak{m}_-) \otimes \overline{V}^\bullet \simeq U_\xi(\mathfrak{m}_-)^* \otimes \overline{V}^\bullet,$$

where \overline{V}^\bullet is an exact complex of vector spaces and the action of $U_\xi(\mathfrak{m}_-)$ on $U_\xi(\mathfrak{m}_-)^*$ is induced by multiplication from the right on $U_\xi(\mathfrak{m}_-)$.

Now by Frobenius reciprocity we have obvious isomorphisms of complexes,

$$\begin{aligned} \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, V^\bullet) &\simeq \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, U_\xi(\mathfrak{m}_-)^* \otimes \overline{V}^\bullet) = \mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, U_\xi(\mathfrak{m}_-)^* \otimes \overline{V}^\bullet) \simeq \\ &\simeq \mathrm{Hom}_{\mathbb{C}}(U_\xi(\mathfrak{m}_-) \otimes_{U_\xi(\mathfrak{m}_-)} \mathbb{C}_\chi, \overline{V}^\bullet) = \overline{V}^\bullet, \end{aligned}$$

where the last complex is exact. Therefore the functor $\mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(Q_\chi, \cdot)$ is exact. \square

The following proposition is an analogue of Theorem 2.3 in [29] for quantum groups at roots of unity.

Proposition 10.19. *Let $\eta \in \mathrm{Spec}(Z_0)$, $\phi \circ \tilde{\pi}(\eta) \in \mu_{M_+}^{-1}(u)$ and assume that a quantum coadjoint transformation $\tilde{g} \in \mathcal{G}$ is fixed as in Theorem 9.12. Denote $\xi = \tilde{g}\eta \in \mathrm{Spec}(Z_0)$ and $d = m^{\dim \mathfrak{m}_-}$. Let χ be a character of $U_\xi(\mathfrak{m}_-)$, \mathbb{C}_χ the corresponding representation of $U_\xi(\mathfrak{m}_-)$. Then $Q_\chi^d \simeq U_\xi(\mathfrak{g})$ as left $U_\xi(\mathfrak{g})$ -modules, $U_\xi(\mathfrak{g}) \simeq \mathrm{Mat}_d(W_\varepsilon^s(G))$ as algebras and $Q_\chi \simeq (W_\varepsilon^s(G)^{opp})^d$ as right $W_\varepsilon^s(G)$ -modules.*

Proof. Let E_i , $i = 1, \dots, S$ be the simple finite-dimensional modules over the finite-dimensional algebra $U_\xi(\mathfrak{g})$. Denote by P_i the projective cover of E_i . Since by Theorem 9.12 the dimension of E_i is divisible by d we have $\dim E_i = dr_i$, $r_i \in \mathbb{N}$, where r_i is the rank of E_i over $U_\xi(\mathfrak{m}_-)$ equal to the dimension of the space of Whittaker vectors in E_i . By Proposition 2.1 in [29]

$$U_\xi(\mathfrak{g}) = \mathrm{Mat}_d(\mathrm{End}_{U_\xi(\mathfrak{g})}(P)^{opp}),$$

where $P = \bigoplus_{i=1}^S P_i^{r_i}$. Therefore to prove the second statement of the proposition it suffices to show that $P \simeq Q$. Since by the previous lemma Q_χ is projective we only need to verify that

$$r_i = \dim \mathrm{Hom}_{U_\xi(\mathfrak{g})}(P, E_i) = \dim \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, E_i).$$

Indeed, by Frobenius reciprocity we have

$$\dim \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, E_i) = \dim \mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, E_i) = r_i.$$

This proves the second statement of the proposition. From Proposition 2.1 in [29] we also deduce that $P^d \simeq U_\xi(\mathfrak{g})$ as left $U_\xi(\mathfrak{g})$ -modules. Together with the isomorphism $P \simeq Q$ this gives the first statement of the proposition.

Using results of Section 6.4 in [26] and the fact that Q_χ is projective one can find an idempotent $e \in U_\xi(\mathfrak{g})$ such that $Q_\chi \simeq U_\xi(\mathfrak{g})e$ as modules and $W_\varepsilon^s(G) \simeq eU_\xi(\mathfrak{g})e$ as algebras.

By the first two statements of this proposition one can also find idempotents $e = e_1, e_2, \dots, e_d \in U_\xi(\mathfrak{g})$ such that $e_1 + \dots + e_d = 1$, $e_i e_j = 0$ if $i \neq j$ and $e_i U_\xi(\mathfrak{g}) = e U_\xi(\mathfrak{g})$ as right $U_\xi(\mathfrak{g})$ -modules. Therefore $e_i U_\xi(\mathfrak{g})e = e U_\xi(\mathfrak{g})e$ as right $e U_\xi(\mathfrak{g})e$ -modules, and

$$Q_\chi \simeq U_\xi(\mathfrak{g})e = \bigoplus_{i=1}^d e_i U_\xi(\mathfrak{g})e \simeq (e U_\xi(\mathfrak{g})e)^d \simeq (W_\varepsilon^s(G)^{opp})^d$$

as right $W_\varepsilon^s(G)$ -modules. This completes the proof of the proposition \square

Corollary 10.20. *The algebra $W_\varepsilon^s(G)$ is finite-dimensional, and $\dim W_\varepsilon^s(G) = m^{\dim \Sigma_s}$.*

Proof. By Proposition 7.4 $2\dim \mathfrak{m}_- + \dim \Sigma_s = \dim G$. Therefore by the definition of Q_χ we have $\dim Q_\chi = m^{\dim G - \dim \mathfrak{m}_-} = m^{\dim \mathfrak{m}_- + \dim \Sigma_s}$. Finally from the last statement of the previous theorem one obtains that $\dim W_\varepsilon^s(G) = \dim Q_\chi / m^{\dim \mathfrak{m}_-} = m^{\dim \Sigma_s}$. \square

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